Reply to Dr. Mikhail Verbitsky's comments (CC2 and CC3)

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Thank you very much for your further comments, which are very thoughtful and valuable for us. We are glad that we are coming closer to common understanding through discussions.

In your report CC2, you state that "*VCV18 bifurcation points can be described as a timescale matching problem between orbital timescale and orbitally modified intrinsic timescale*". We agree with your point that the range of the intrinsic timescale allowing a particular resonance ($P \sim T$) depends on the amplitude of astronomical forcing, and that such resonances may not occur if the forcing is too weak. On that basis, we enlighten the fact that, in many ice age models under astronomical forcing **with a realistic amplitude**, the ∼100-kyr responses arise if the model's intrinsic timescale is close to ∼100 kyr. That is, our conclusion sustains for the realistic amplitude of the astronomical forcing: $A \approx 1$ in our terminology and $\varepsilon \approx 1$ in VCV18's term.

Inspired by your scaling analysis, we propose the following physical argument. It does not use the Pi-theorem but as we show next it converges to a conclusion similar to yours. Following the VCV18 paper, the height of the fully developed ice sheet is given by $H = \zeta S_0^{1/4}$ and the snow accumulation rate is a. Using your comment (CC1), the intrinsic time scale of advection **in the absence of forcing** is given as

$$\tau_{adv} = \frac{H}{a} = \frac{\zeta S_0^{1/4}}{a}.$$

Since the snow accumulation rate a and the forcing term $\varepsilon F_S(t)$ appear as $a - \varepsilon F_S(t)$ in the dynamical equations of VCV18, we assume that the **net ice accumulation rate under a forcing cycle** scales as $a - c\varepsilon$: this is similar to $a - \varepsilon F_S(t)$ but we introduce a cycle-specific coefficient c. Indeed, the astronomical forcing is a complicated signal and its amplitude from cycle to cycle. For this cycle (of period, say, T) to actually entertain a resonance with glaciation dynamics, we expect the typical ice build-up time to match T, i.e.,

$$\text{(glaciation period)} = \frac{\text{(maximal ice-sheet height)}}{\text{(net ice accumulation rate)}} = \frac{H}{a - c\varepsilon} \sim T.$$

$$\Leftrightarrow \; c \sim \frac{1}{\varepsilon}\left(a - \frac{H}{T}\right) = \frac{a}{\varepsilon T}\left(T - \tau_{adv}\right).$$

This equation must hold for a majority of cycles, that is, for a range of c denoted by $-c_1 < c < c_2$ ($c_1, c_2 > 0$). Thus,

$$-c_1 \lesssim \frac{a}{\varepsilon T}\left(T - \tau_{adv}\right) \lesssim c_2$$

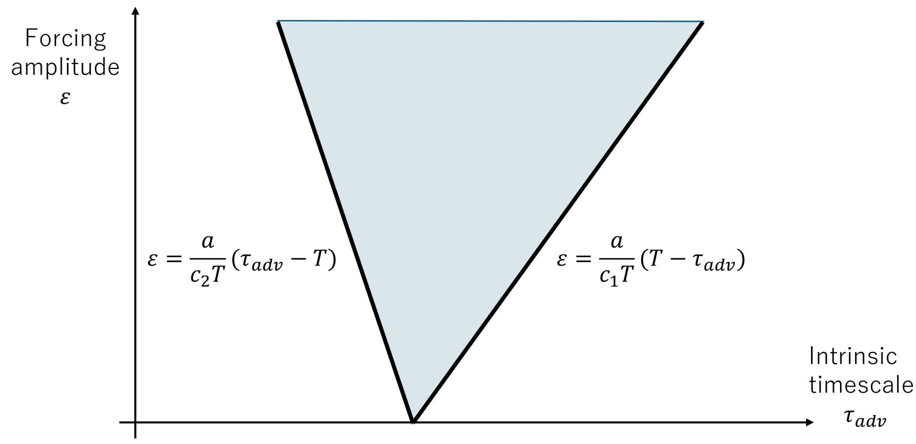

Figure 1. The parameter region derived from the simple physical consideration. The resonance with the astronomical period T is possible at least within the triangular region.

That is,

$$\varepsilon \gtrsim \frac{a}{c_1 T}\left(\tau_{adv} - T\right) \text{ and } \varepsilon \gtrsim \frac{a}{c_2 T}\left(T - \tau_{adv}\right).$$

These inequalities imply a triangular region in τ_{adv}–ε space (Fig. 1 here). The system may resonate at the astronomical period T at least within the triangular region. If we interpret τ_{adv} as the intrinsic timescale of the system and if use the notation of our article ($\tau_{adv} = rT_0$ and $\varepsilon = A$), the above inequalities are

$$A \gtrsim \frac{a}{c_1 T}\left(rT_0 - T\right) \text{ and } A \gtrsim \frac{a}{c_2 T}\left(T - rT_0\right).$$

The resonance may occur at least within this region, but the actual resonance region is more complicated than suggested from the above equations because of nonlinear effects (cf. Figs. 4 and S5 in our article).

The inequalities derived here are slightly different from what you drive using a scaling analysis in CC2. However, we reach essentially the same conclusion that the range of the intrinsic timescale leading to a particular resonance ($P \sim T$) must depend on the forcing amplitude if the forcing amplitude changes significantly. Our conclusion holds for the astronomical forcing with realistic amplitude. **This point will be addressed in the revised manuscript.**

We would like to thank you again for guiding us to the physical considerations.