Reply to Dr. Mikhail Verbitsky's comments

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Thank you very much for thoroughly reviewing our manuscript and providing very valuable feedback. Below, we reply to your comments (shown in *italic*) and propose several changes to the manuscript (shown in **bold**). We believe that these revisions will significantly enhance the quality and clarity of our work.

Main point

In your report, you argue that *astronomical forcing makes the intrinsic timescale irrelevant* and hence *the intrinsic timescale has no role in the present results* (pp. 3–4). We respectfully disagree with these statements. Your argument is based on a scaling analysis. Using Buckingham's π-theorem, the system's response period P is expressed as shown in your Eq. (8):

$$\frac{P}{\tau_{\text{int}}} = \Phi\left(\frac{\varepsilon}{a}, \frac{T}{\tau_{\text{int}}}, V\right), \tag{8}$$

where τ_{int} is the system's intrinsic timescale, ε is the amplitude of the forcing, a is the mass influx to the ice sheets, T is a period of astronomical forcing, and V is the parameter controlling the balance between positive and negative feedbacks. We agree with Eq. (8) itself, but you continue with the assertion that *we know from experiments with VCV model ... that for T =35–50 kyr ... the system responds with the period-doubling. This means that $\frac{P}{\tau_{int}}$ depends linearly on T, i.e.,*

$$\frac{P}{\tau_{int}} = \frac{T}{\tau_{int}}\Phi\left(\frac{\varepsilon}{a}, V\right),\ \textit{or}\ \frac{P}{T} = \Phi\left(\frac{\varepsilon}{a}, V\right), \tag{9}$$

we can see that astronomical forcing makes the intrinsic timescale irrelevant. However, Eq. (9) is only locally true in the parameter space because $\frac{P}{\tau_{int}}$ depends nonlinearly on $\frac{T}{\tau_{int}}$ across different modes of resonances and non-resonances. This is shown in our Fig. 4b as a nonlinear dependence of P on τ_{int} (rT_0 in our case). Therefore, we sustain our conclusion that the intrinsic timescale of the system is indeed critical for realizing the 100 kyr response. Of course, in each resonance mode, P is fixed to T or some combination of Ts, consistently to your argument. Thus, in terms of π-theorem, Eq. (8) is a piecewise linear function of T, whose discontinuous points are determined by τ_{int}. **In order to avoid this confusion, we clarify in the conclusion why the intrinsic timescale τ_{int} is relevant with the system's response period P, while P locally obeys to one of astronomical period T.**

Other confusions to be clarified

- We have not mentioned that the system's response period to the astronomical forcing *is independent of the amplitude of the astronomical forcing* (p. 1 in your report). Instead, in lines 214–216, we have mentioned that the natural periodicity leading to 95-kyr cycles shifts toward larger values as the amplitude of the astronomical forcing increases. Nevertheless, for realistic forcing amplitudes $A \approx 1$, only natural periodicities near 95-kyr, specifically $83 \le rT_0 \le 118$ kyr, allows to resonate with 95 kyr astronomical cycles (Fig. 4b). **In order to avoid the confusion, we mention, in the conclusion, that while the intrinsic periodicity being close to $\sim$100-kyr is necessary to realize $\sim$100-kyr cycles, the amplitude of astronomical forcing (or the sensitivity to it) is also a factor determining the response frequency.**

- You stress that *there is no similarity between ice sheets with and without forcing* (your Section 4). We also consider that ice-sheet dynamics with and without forcing are qualitatively different. In the case of the VCV18 model, the dynamics under weak focing is close to a linear response to the obliquity cycles, while the dynamics under strong forcing is characterized by nonlinear resonance at $\sim$100-kyr time scales. Our understanding is that the concept of nonlinear resonance does not entail a physical similarity between the forced and unforced systems: in general nonlinear resonance, the resonance frequency ω under the forcing and the natural frequency ω_0 in the absence of forcing can differ. However, in standard cases, the nonlinear resonance frequency ω shifts continuously from the natural frequency ω_0, typically following $\omega(A) = \omega_0 + \kappa A^2$, where κ is a constant defined by the coefficient of nonlinear restoring force and the natural frequency (https://en.wikipedia.org/wiki/Nonlinear_resonance). Therefore, it is reasonable to assume that the nonlinear resonance frequency is not too far away from the natural frequency of unforced system as long as the forcing amplitude is moderate. **In the revised paper, we address the underlying assumption for discussing the nonlinear resonance in the VCV18 model based on its natural frequency in the absence of forcing.**

- The natural periodicity of the VCV18 model was calculated to be 95 kyr in our work. This value is coincidentally identical to the observed principal period of ice age cycles as well as one of the eccentricity periods. However, we do not need this coincidence for our conclusion. The natural periodicity does not have to be sharply at 95 kyr for realizing the 95 kyr cycles. Indeed, the resonance at 95 kyr can occur for a range of natural periodicities, $83 \le rT_0 \le 118$ kyr for the realistic astronomical forcing (see line 219). **In the revised paper, we will mention that the exact numerical match between the natural periodicity and the 95 kyr eccentricity periodicity is purely coincidental and unnecessary to achieve the 95 kyr resonance.**

Finally, we would like to thank you again. A couple of equations introduced in your report help theoretical considerations. While your comments are critical, we believe that apparent contradictions between your opinions and our thoughts can be solved by careful clarifications proposed above.