Response to Referee 1

Rojo-Garibladi, Contreras–López; Giannerini; Salas–de–León; Vázquez–Guerra; Cartwright

August 21, 2023

Dear Referee,

many thanks for your comments, please find below a pointwise reply to all of them, where your comments are in italic.

1. *In the introduction authors ought to add some considerations on the impact of El Niño on human life, and, in particular, on human health. This would enhance the relevance of their work and increase its readership.*

   We propose a modification to the paragraph, adding the following:

   In a context of global warming, it is important to understand the spatial and temporal patterns of atmospheric temperatures in complex systems like the Eastern South Pacific region. Gaining knowledge on temperature changes, and in the phenomena behind them, is essential for understanding both scientific and societal issues, since El Niño events have a great impact on human life [Glantz, 2022], and, in particular, on human health [Kovats et al., 2003] far beyond the Eastern South Pacific region [Fan et al., 2017].

2. *Section 2.1.2 needs more bibliography.*

   thank you, we have added more bibliography.

3. *Section 2.1.2: the final sentence on Friedrich et al must be anticipated and also expanded: it is not good to say the readers: go and read that paper. Moreover, the journal is on a topic that is very different from the one of this journal, thus maybe readers could incur in some difficulties in reading that paper. For example, in section 2.1.3 authors succintlu but clearly describe the main content of various papers*

   Thank you for the comment, we propose to add the following at the beginning of Section 2.1.2:

   > One recurring issue with trend estimation is that, in most situations, the detrended series is both dependent and possibly heteroskedastic; moreover, missing data are very common and disregarding these aspects leads to invalid confidence bands. Here, we follow Friedrich et al. [2020], that solve the problem by proposing a novel autoregressive wild bootstrap scheme that does not need adjustments in the presence of missing data and results particularly suitable for climatological applications. We assume that the series X_t, $t = 1, \dots, n$ admits the following decomposition ...

4. *Authors explain that they use "a neural network model of the map F and of its Jacobian J". This is a key point of the whole paper and, in particular, of sections 2.2 and 3. In section 3 we also read that the ANN has a single layer. Authors must provide further details, such has the number of neurons of the layer, the training procedure, the global minimization algorithm used and the activation functions.*

   Thank you for the comment, we have reworked the main paragraph of section 2.2 where we explain in more detail the technical aspects. Moreover, we have added the following section in the Appendix, providing further details on neural networks. This in order not to burden the main text with too much technical information, and, at the same time, provide the necessary level of detail to the interested reader:

A Neural Network Models for Random Dynamical Systems

As in Section 2.2, the dynamical system has the following representation:

$$\mathbf{X}_{t+1} = F(\mathbf{X}_t) + \mathbf{E}_{t+1}, \qquad \mathbf{X}_t \in \mathbb{R}^d, \tag{1}$$

where $\mathbf{E}_t$ is an error process in $\mathbb{R}^d$. We assume that the data $\{X_t\}$ are generated by the nonlinear autoregressive model

$$X_{t+1} = f_d(X_t, X_{t-1}, \ldots, X_{t-d+1}) + \varepsilon_t, \tag{2}$$

where $X_t \in \mathbb{R}$ and $\{\varepsilon_t\}$ is a sequence of independent random variables with $E(\varepsilon_t) = 0$ and $\mathrm{Var}(\varepsilon_t) = \sigma^2$. Also, we denote with $\mathbf{J}_t$ the Jacobian of the map, evaluated at $\mathbf{X}_t$. The model of Eq. (1) can be seen as the state–space representation of the system of Eq. (2), where $\mathbf{X}_t = (X_t, X_{t-1}, \ldots, X_{t-d+1})$ and $\mathbf{E}_t = (\varepsilon_t, 0, 0, \ldots, 0)$. We derive a consistent estimator for the map F and its Jacobian through a neural networks estimator of f_d. "Neural Networks" are a class of nonlinear models inspired by the neural architecture of the brain [Nychka et al., 1992]. These are made up of layers that in turn have connected "neurons", which send messages and share information between each other. Layers are classified into three groups: 1) input, 2) hidden, and 3) output. The input values $\mathbf{X}_t$ are received by the *input units*, which simply pass the input forward to the hidden units u_j. Each connection (indicated by an arrow) performs a linear transformation determined by the *connection strength* ω_{ij} so that the total input to unit u_j results

$$\sum_{i=1}^{d} \omega_{ij} X_{t-i+1} + \omega_{0j} \tag{3}$$

and each unit performs a nonlinear transformation on its total input:

$$u_j = \psi\left(\sum_{i=1}^{d} \omega_{ij} X_{t-i+1} + \omega_{0j}\right). \tag{4}$$

The *activation function* ψ is sigmoidal function with limiting values 0 and 1 as $x \to -\infty$ and $+\infty$, respectively. Here we adopt the following:

$$\psi(x) = \frac{x(1 + |\frac{x}{2}|)}{2 + |x| + \frac{x^2}{2}} \tag{5}$$

The hidden layer outputs u_j are passed along to the *single output unit*, which performs an affine transformation on its total input. Therefore, the network output f_d, for d inputs and h units in the hidden layer, can be represented as:

$$\begin{aligned} f_d(\mathbf{X}_t) = f_d(X_t, X_{t-1}, \ldots, X_{t-d+1}) &= \beta_0 + \sum_{j=1}^{h} \beta_j u_j \\ &= \beta_0 + \sum_{j=1}^{h} \beta_j \psi\left(\sum_{i=1}^{d} \omega_{ij} X_{t-i+1} + \omega_{0j}\right). \end{aligned}$$

We made use of the R `nnet` package to implement the estimator and we used Least Squares minimization [Venables and Ripley, 2002]. We also experimented with conditional maximum likelihood without noticing major discrepancies in the results. A practical difficulty in regression with neural network models is selecting among the many possible combinations of d and h. Here we choose the best model that minimises the BIC defined as:

$$\mathrm{BIC} = \log(\hat{\sigma}^2) + \frac{\log(n)}{n}[1 + h(d+2)], \tag{6}$$

where the error variance is estimated through the residual sum of squares $\mathrm{RSS} = n^{-1}\sum_{t=1}^{n}(X_t - \hat{f}_d(\mathbf{X}_{t-1}))^2$. Once the estimator $\hat{f}_d$ is obtained, a consistent estimator for the map F and its Jacobian $\mathbf{J}$ can be derived by plug-in methods, as described in [Shintani and Linton, 2004].

5. *A lot of important details are in the appendix, especially in appendix A5. Authors ought to consider to move some materials of the appendix in the full text*

   We can look at moving some of A5 to the main text, but we are not sure that we will end up moving it. We already spent quite a long time thinking about what should go into the main text and appendix. We put into the main text what we thought to be essential and into the appendix information that's useful for those who want it but not essential for a general reader.

References

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