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Origins and suppression of bifurcation phenomena in lower-order monsoon models

S. Krishna Kumar[1] and Ashwin K Seshadri[1,2]

[1]Divecha Centre for Climate Change, Indian Institute of Science, Bengaluru, India
[2]Centre for Atmospheric and Oceanic Sciences, Indian Institute of Science, Bengaluru, India

Correspondence: S. Krishna Kumar (mailbox.skk@gmail.com)

Abstract. South Asian monsoon rainfall varies rapidly in the paleoclimate record, and this has been interpreted using simple models as arising from tipping points. This study explores a class of simple monsoon models, based on convective quasi-equilibrium, and the bifurcations permitted by their mathematical forms. Specifically, low-order models are derived starting from the Quasi-equilibrium tropical circulation model (QTCM) to examine the bifurcations present. Previous studies that have pointed to an abrupt transition in low-order monsoon models typically identify a saddle node bifurcation occurring as a result of changes in the radiation budget. The present study shows how such saddle node structures arise across a wide range of modeling assumptions and parameter values, and yet permit a continuous transition into and out of precipitating regimes without any bifurcation being physically manifest. This is because the bifurcation points lie in a regime that is not physically relevant when the dry thermal stratification is sufficiently large. As a result, these low-order models can be interpreted as possessing abrupt transitions that are latent in the equations but do not express themselves physically. However, when the dry thermal stratification is reduced, bifurcations can occur. This paper also shows that these latent saddle-node structures are themselves part of the unfolding of a pitchfork bifurcation. These findings help understand the role of stabilizing phenomena on the general absence of abrupt monsoon transitions despite the presence of nonlinear terms in these models.

1 Introduction

The South Asian monsoon is an extensively studied climatic phenomenon, important for the large and wide-ranging socioeconomic impact of monsoon rainfall for the region and its inhabitants (Gadgil and Gadgil, 2006). The monsoon itself is influenced by broader climatic phenomena, from the El-Nino Southern Oscillation (ENSO) in the Pacific (Webster et al., 1998), north Atlantic sea surface temperatures (Goswami et al., 2006; Kucharski et al., 2008; Pottapinjara et al., 2014), to oscillations in the Indian ocean (Saji et al., 1999; Ashok et al., 2001). Fully coupled atmosphere-ocean general circulation models (AOGCMs) are necessary to render the multi-scale nature of these interactions between the monsoon and rest of the climate system, as well as to simulate variability and change (Held, 2005; Zhou and Xie, 2018; Schneider and Dickinson, 1974). However, it is increasingly recognized that understanding of these complex systems is aided by understanding the relationships among a range of models of varying complexity, with model hierarchies being central to this effort (Held, 2005; Jeevanjee et al., 2017; Maher et al., 2019; Polvani et al., 2017). For example, AOGCMs are perhaps too complex to permit isolation of the mecha-

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nisms that might give rise to bifurcation phenomena in monsoons. Lower order conceptual models, on the other hand, involve many simplifying assumptions and usually highlight particular facets of the overall dynamics, focusing on a small subset of the wide range of scales that are typically involved (Held, 2005; Palmer, 2019). Still, these low-order models can be useful if they help identify the roles of particular processes. Thus, in case of monsoons, if few processes give rise to nonlinear phenomena (such as bifurcations) low order models that isolate these processes can still be useful (Ghil, 1989). Between the two extremes, models of intermediate complexity such as the Quasi-equilibrium Tropical Circulation model (QTCM) (Neelin and Zeng, 2000; Zeng et al., 2000; Bellon and Sobel, 2008b, a, 2010; Burns et al., 2006), offer a middle ground, making simplifications to the fully three-dimensional equations describing the atmosphere, while retaining the horizontal structure that yields the diversity of phenomena that persist once the convective quasi-equilibrium assumption is made. Moreover, such simplified albeit infinite-dimensional models such as QTCM can serve as a basis for developing further low-order models, such as the ones examined in this paper.

This study examines the nonlinear dynamics of lower order models derived from the convective-equilibrium framework, in particular the QTCM model, focusing on the behavior of steady-states of these models and the bifurcations they can admit. An important simplification of QTCM1 to study monsoons (Boos and Storelvmo, 2016a) yields a set of three equations, describing the interactions of the first baroclinic modes of velocity, temperature and moisture, which are advected from ocean towards a tropical continent. The authors (Boos and Storelvmo, 2016a) used the model to examine whether the seasonal mean monsoon yielded an abrupt response to the change in the forcing from solar radiation. Furthermore, they noted that any mechanism facilitating such an abrupt response might also explain the sudden onset of the Indian summer monsoon as the seasonal cycle of insolation unfolds. Considering the steady-state, approximating the seasonal mean monsoon, the study (Boos and Storelvmo, 2016a) found a single stable equilibrium in the mean monsoonal velocity, which did not display an abrupt transition as the parameter describing solar insolation was varied. The monsoonal regime, with landward winds in the boundary layer and nonzero precipitation, was present if the source of energy to the atmospheric column was positive. The non-monsoonal regime had zero precipitation and oceanward winds in the boundary layer, corresponding to a negative column energy source. Transitions between these two regimes were continuous as the monsoonal regime shifted continuously from the precipitating to nonprecipitating regime, and hence the monsoon's response to changing energy source was found to be near-linear. However if in the model, the term accounting for dry thermal stratification was neglected, the system exhibited an abrupt transition between these regimes as a saddle-node bifurcation became manifest in the monsoonal regime at positive values of precipitation and winds (Boos and Storelvmo, 2016a, b).

The present understanding of this abrupt change in such models is that it is unphysical, being an artefact of neglecting dry thermal stratification (Boos and Storelvmo, 2016a, b; Seshadri, 2017). This stratification, arising from the increase in dry static stability with height, is the dominant contribution in the thermodynamic balance of monsoons. The thermal stratification counters the destabilizing nature of diabatic heating in the column, and comes from the fact that the dry static energy, and consequently the moist static energy also, increases with height. Thus, the stratification was interpreted to add to the stability of the system, and the resulting gross moist stability was positive, though small (Boos and Storelvmo, 2016a).

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It is to be noted that rapid changes are often apparent in the paleoclimatic evidence (Prell and Kutzbach, 1987; Jalihal et al., 2019) but it is not known whether these transitions correspond to tipping behavior. Furthermore, while the rapid onset of monsoons as compared to the gradual progression of the seasonal cycle of solar insolation has intriguingly pointed to the role of nonlinear processes (Boos and Storelvmo, 2016a), and several studies have considered the dynamical aspects that could play a role, there is no conclusive account of a bifurcation origin for onset (Schneider and Bordoni, 2008; Bordoni and Schneider, 2008). As we noted earlier, Boos and Storelvmo (2016a) reported that a bifurcation was seen in their model only when a dominant term in the thermodynamic balance, the dry thermal stratification, was ignored. Consistent with this result is the study of Seshadri (2017), which pointed out that bifurcation arose in the simple models of (Levermann et al., 2009) owing to the neglect of the static stability of troposphere. Seshadri (2017) showed that dry thermal stratification provided a stabilizing effect in the energy balance, which exported dry static energy through a term that is linear in the wind-speed. In contrast, the stabilizing effect in the absence of this stratification is much weaker, relying on horizontal advection and thus depending quadratically on wind-speed. Since the slope of quadratic terms is prone to become negligible as the circulation weakens, a bifurcation arose as an intrinsic part of models neglecting this thermal stratification. In the presence of thermal stratification, its presence countered the destabilising effects of radiation and diabatic heating during monsoons. The study concluded that bifurcation is not intrinsic to models in the presence of dry thermal stratification, and is unlikely to be present, but can arise in a less probable region of parameter space wherein the stabilizing effect is weak (Seshadri, 2017).

This study extends these early investigations of monsoon bifurcations, through detailed analysis of low order models that are based on QTCM. While low-order monsoon models have been investigated to study whether they contain bifurcations, their mathematical structure and range of bifurcations merit detailed study. The studies of Boos and Storelvmo (2016a) and Seshadri (2017) illustrate that a model's being nonlinear does not ensure that it will contain bifurcations. At the same time nonlinear advection, whose effects motivated the account of Levermann et al. (2009), is expected to give rise to rich phenomena including bifurcations. It is the goal of this paper to reconcile these apparently contradictory facts. Section 2 explains the QTCM and simplifications undertaken to obtain low-order models. Section 3.1 revisits the analysis and findings of Boos and Storelvmo (2016a) and identifies a saddle node complex, which is the underlying phenomenon arising in the low order models derived from QTCM, in cases of bifurcating as well as non-bifurcating behaviours. Section 3.2 describes in detail the stabilising role of stratification in light of the saddle node complexes that exist and confirms that reducing the magnitude of the dry thermal stratification makes a bifurcation more likely. Section 3.3 explores these results further and shows that the saddle node complex themselves arise from unfolding of a pitchfork bifurcation. Implications of these results for lower order models of monsoons are further examined in Section 4.

2 Background & Methods

The quasi-equilibrium tropical circulation model (QTCM) uses tailored basis functions to describe the vertical profiles of temperature, moisture, and winds, under the assumption of convective quasi-equilibrium (Neelin and Zeng, 2000). The framework assumes that there is a separation of time-scales between the removal of instability through convection, which occurs very

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Table 1. Variables and parameters used in the simplified model based on QTCM.

Notation	Quantity
T_{1L}	Temperature basis associated with baroclinic wind for land boundary (in $\mathrm{J/kg}$)
q_{1L}	Moisture basis associated with baroclinic wind for land boundary (in $\mathrm{J/kg}$)
v_{1L}	Meridional component of baroclinic wind for land boundary (in $\mathrm{m/s}$)
T_{1s}	Temperature basis associated with baroclinic wind for sea boundary (in $\mathrm{J/kg}$)
q_{1s}	Moisture basis associated with baroclinic wind for sea boundary (in $\mathrm{J/kg}$)
v_{1s}	Meridional component of baroclinic wind for sea boundary (in $\mathrm{J/kg}$)
ϵ_1	Coefficient of vertical momentum transfer for baroclinic wind or inverse of mechanical damping time scale (per second)
κ	Poisson constant (ratio of gas constant to specific heat at constant pressure for dry air)
L	Horizontal length scale for the column of atmosphere (in m)
τ_c	Convective adjustment time (in seconds)
M_{sr}	Reference gross dry stability (thermal stratification, in $\mathrm{J/kg}$)
M_{sp}	Change in gross dry stability per T_1 change (non-dimensional)
M_{qr}	Reference value of gross moisture stratification (in $\mathrm{J/kg}$)
M_{qp}	Change in gross moisture stratification per q_1 change (non-dimensional)
$\langle a_1 v_1 \rangle$	Temperature advection coefficient associated with baroclinic wind (vertical integral of product of temperature basis function a_1 and velocity basis function v_1)
$\langle b_1 v_1 \rangle$	Moisture advection coefficient associated with baroclinic wind (vertical integral of product of moisture basis function b_1 and velocity basis function v_1)
E	Surface evaporation (in $\mathrm{J/kg.s}$)
H	Surface sensible heat flux (in $\mathrm{J/kg.s}$)
R	Atmospheric radiative flux convergence (in $\mathrm{J/kg.s}$)
g	Gravitational acceleration (in $\mathrm{m/s^2}$)
p_t	Reference pressure depth of troposphere (in Pa)

rapidly as compared to the large-scale dynamics. Thus, there is a reduction in the degrees of freedom of the model, whose vertical structure is governed by its being approximately in convective equilibrium (Neelin and Zeng, 2000; Zeng et al., 2000). Such an approach essentially has convective processes being strongly coupled to large-scale dynamics, with the resulting framework providing for a sequence of intermediate models with successive degrees of completeness (Neelin and Zeng, 2000). The simplest formulation, called QTCM1 assumes a single vertical basis function derived from the convective equilibrium profile of temperature (Zeng et al., 2000).

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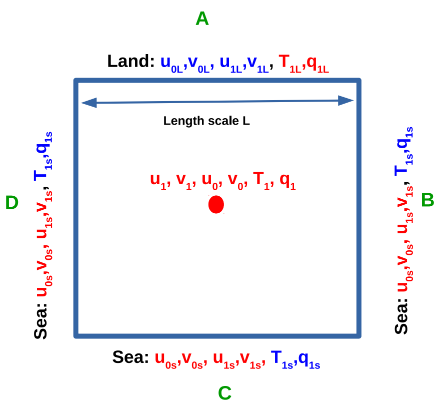
In QTCM, the primitive equations describing the evolution of momentum, thermodynamic and moisture fields are approximated for the strongly convective regions and the resulting analytical solutions are embedded in the model's numerical scheme to be applied even to those regions that are not necessarily in convective equilibrium. This is an important simplification of the model, and leads to a set of vertically averaged partial differential equations (PDEs), which are obtained by projecting the primitive equations onto the vertical basis functions (see Supplementary Information, Fig. $S1$). The basis functions in turn describe the barotropic and baroclinic modes of the zonal and meridional velocity components, along with associated temperature and moisture fields.



u_{nB}, v_{nB}	Zonal (u) and meridional (v) components of barotropic ($n = 0$) / baroclinic ($n = 1$) velocity over land ($B \equiv L$)/ sea ($B \equiv s$) boundary
T_{1B}, q_{1B}	Temperature (T) and moisture (q) in energy units for land ($B \equiv L$)/ sea ($B \equiv s$)

Figure 1. (b)Schematic of finite difference scheme for a square column of side L and boundaries A,B,C,D. For this study, the Land/Sea boundary conditions on four sides are 'LSSS' clockwise from top, with the unknown variables in the column being interpolated from the boundary values. The prescribed variables are shown in blue and the unknown variables are shown in red.

While QTCM by itself exhibits a reduction of mathematical complexity as compared to the three-dimensional primitive equations, it can be useful for obtaining yet simpler reduced order models too. Moreover, such models which embody convective closures that are based on a nearly moist-adiabatic atmosphere simplify the effort to obtain ever more simplified low-order models that arise from this dynamics. There are many ways to go from PDE-based models such as QTCM to finite-dimensional ODE based systems. Galerkin-based spectral methods that further project the equations onto horizontal basis functions are one such approach. We do not take this approach, but instead reduce the set of PDEs (see Supplementary Information) to a set of ODEs using finite differences in order to approximate horizontal derivatives, following the approach of Boos and Storelvmo (2016a). For generation of simple models, as in Boos and Storelvmo (2016a), a single tropospheric column or box over the relevant landmass can be treated along with the conservation of quantities within it. At surface level, the column is supposed to be bounded on its four sides (sides A-D in Fig. 1) either by land or by sea. Replacing partial derivatives in the QTCM equations with corresponding finite differences estimated over this box yields a system of ODEs. Thus, one could model, say, the Indian peninsula as a land column with land on one boundary and ocean/sea on the other three. For sake of convenience, this configuration is labelled as LSSS, with the order of boundaries being clockwise, starting from the top or north. With additional

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simplifying assumptions relevant to the chosen region, a set of algebraic equations governing equilibrium solutions for the system can be obtained, as in Boos and Storelvmo (2016a) (see Supplementary Information). For the assumptions considered in the model of (Boos and Storelvmo, 2016a), the LSSS case effectively reduces to the Land-Ocean formulation in Boos and Storelvmo (2016a). This is the approach that we shall follow, yielding a range of simple models.

In QTCM, precipitation affects the thermodynamic equation and moisture equation through convective heating and moistening terms respectively, and its closure in the model can affect the overall dynamics. An often used closure has precipitation P taken to be directly proportional to $q_1 - T_1$, where q_1 and T_1 are the first terms in the respective expansions of moisture q and temperature T, both of which are expressed in energy units in QTCM.

$$P = \frac{1}{\tau_c}(q_1 - T_1)\mathcal{H}(q_1 - T_1) \tag{1}$$

where $\mathcal{H}(q_1 - T_1)$ is the Heaviside function, which ensures that precipitation is non-negative. Here, τ_c is the convective time scale, over which convection causes the troposphere to move towards a equilibrium moist adiabatic state where $q_1 = T_1$ in energy units. When $T_1 > q_1$, the dry and warm column is stable against moist convection and no precipitation is expected to occur, leading to $P = 0$. The implication of this closure is that a warmer column without concomitant increase in moisture tends to be stable to moist convection.

Such a precipitation formulation also underpins analysis of the system behaviour in two parts. The simple lower order model obtained from QTCM1 is examined in two distinct regimes: a case where the precipitation is positive (precipitating regime), and a case where precipitation is zero (non-precipitating regime), *i.e.*, $q_1 - T_1 \leq 0$. The consideration of these two cases separately means that the QTCM1 equations are reduced to two distinct sets of ODEs using finite differences, one for the $P = 0$ case and the second for the non-zero case (See Supplementary Information). These two sets are identical at $P = 0$, elsewhere they differ. Note that, the overall model follows each of these sets of equations in the corresponding regime, implied concisely by the Heaviside function in the precipitation expression. Here, the emphasis is to understand the response of these two sets of ODEs separately and describe the behaviour of the overall model in terms of their composite behavior. Further, it is to be noted that the overall precipitation is not represented here as a superposition or linear combination of the $P > 0$ and $P = 0$ cases. Instead, the use of Heaviside function just means that the two cases have their own regions of relevance in the $v_{1s} - R$ space, *i.e.,* the variable-parameter space. The simplified system for the non-zero precipitation case is:

$$\frac{\mathrm{d}v_{1s}}{\mathrm{d}t} + \epsilon_1 v_{1s} + (T_{1L}\kappa)/L - (T_{1s}\kappa)/L = 0 \quad . \tag{2a}$$

$$\frac{\mathrm{d}T_{1L}}{\mathrm{d}t} + T_{1L}/\tau_c - q_{1L}/\tau_c - (M_{sr}v_{1s})/L - (Hg)/p_t - (M_{sp}T_{1L}v_{1s})/L - (Rg)/p_t + \tag{2b}$$
$$(T_{1L}\langle a_1 v_1\rangle v_{1s})/L - (T_{1s}\langle a_1 v_1\rangle v_{1s})/L = 0.$$

$$\frac{\mathrm{d}q_{1L}}{\mathrm{d}t} + q_{1L}/\tau_c - T_{1L}/\tau_c + (M_{qr}v_{1s})/L - (Eg)/p_t + (M_{qp}q_{1L}v_{1s})/L + \tag{2c}$$
$$(\langle b_1 v_1\rangle q_{1L}v_{1s})/L - (\langle b_1 v_1\rangle q_{1s}v_{1s})/L = 0.$$

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Note that the spatial derivatives have been replaced by the corresponding finite difference approximations. The parameters and variables are defined in Table 1. The simplified model for the $P = 0$ case is :

$$\frac{\mathrm{d}v_{1s}}{\mathrm{d}t} + \epsilon_1 v_{1s} + (T_{1L}\kappa)/L - (T_{1s}\kappa)/L = 0. \tag{3a}$$

$$\frac{\mathrm{d}T_{1L}}{\mathrm{d}t} + T_{1L}\langle a_1 v_1\rangle v_{1s})/L - (Hg)/p_t - (M_{sp}T_{1L}v_{1s})/L - (Rg)/p_t - \tag{3b}$$

$$(M_{sr}v_{1s})/L - (T_{1s}\langle a_1 v_1\rangle v_{1s})/L = 0.$$

$$\frac{\mathrm{d}q_{1L}}{\mathrm{d}t} + (M_{qr}v_{1s})/L - (Eg)/p_t + (M_{qp}q_{1L}v_{1s})/L + (\langle b_1 v_1\rangle q_{1L}v_{1s})/L - (\langle b_1 v_1\rangle q_{1s}v_{1s})/L = 0 \quad . \tag{3c}$$

To find the equilibria and their stability for the model system, the time derivatives are set to zero, yielding a single algebraic equation for the equilibria in terms of a choice of one of the three state variables such as, for example, the meridional velocity. The resulting equilibrium equation is a cubic one for the $P > 0$ case and a quadratic one for the $P = 0$ case (see Equations 4 and 5). We recall that the simplifications describe a non-rotational, predominantly meridional and baroclinic flow, confined to tropical latitudes.

The form of the resulting equations for the equilibria, in terms of the meridional velocity component v_1 for the $P > 0$ case is,

$$A_c v_{1s}^3 + B_c v_{1s}^2 + C_c v_{1s} + D_c = 0, \tag{4}$$

where,

$$A_c = (M_{sp} - \langle a_1 v_1\rangle)(M_{qp} + \langle b_1 v_1\rangle)L\epsilon_1 p_t \tau_c,$$

$$B_c = -(M_{sp} - \langle a_1 v_1\rangle - M_{qp} - \langle b_1 v_1\rangle)\epsilon_1 L^2 p_t + (-M_{sr}M_{qp} - M_{sr}\langle b_1 v_1\rangle$$

$$-M_{sp}M_{qp}T_{1s} - M_{sp}T_{1s}\langle b_1 v_1\rangle)\kappa p_t,$$

$$C_c = (-M_{sr} - M_{sp}T_{1s} - \langle b_1 v_1\rangle q_{1s} + M_{qr})L\kappa p_t - (RM_{qp} + R\langle b_1 v_1\rangle +$$

$$HM_{qp} + H\langle b_1 v_1\rangle)g\tau_c\kappa L + (M_{qp} + \langle b_1 v_1\rangle)LT_{1s}\kappa p_t,$$

$$D_c = -(R + H + E)gL^2\kappa.$$

This is a non-homogenous cubic equation. The cubic coefficient essentially reflects the competition between the advection and stratification terms in the thermodynamic and moisture equations. Also note that the radiative forcing R affects both the zeroth order and the linear terms. The corresponding equilibrium equation for the case of $P = 0$ is

$$B_q v_{1s}^2 + C_q v_{1s} + D_q = 0, \tag{5}$$

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where,

$B_q = (\langle a_1 v_1 \rangle - M_{sp})\epsilon_1,$

$C_q = (M_{sr} + M_{sp}T_{1s})\kappa/L,$

$D_q = (H + R)g\kappa/p_t = 0.$

This equation is quadratic, and setting $P = 0$ essentially decouples the temperature and the moisture fields. The moisture variable is now a function of v_{1s} and is in fact a diagnostic quantity (see Eq. 2b, with $q_{1L} = T_{1L}$ resulting from $P = 0$), since it has no influence on the evolution of the other variables. The system, with respect to the computation of equilibria, is $2-d$ in nature with the velocity equation and the thermodynamic equation being coupled to each other. The nonlinear effects of moisture advection and convergence are removed from the overall dynamics of the system, and the decoupling of the moisture variable from temperature leads to the nonlinearity being one order smaller for the equilibrium equation (also see Supplementary Information).

3 Results

3.1 A Complex of saddle node curves

As noted above, there are two different systems of ODEs, for the two cases where $P > 0$ and $P = 0$. The equilibrium equations for these respective systems are separately examined and bifurcation diagrams plotted (Fig.2). This figure depicts the response of the model for the standard set of parameters used in Boos and Storelvmo (2016a). For both the cases, distinction is made between stable and unstable equilibrium branches, and furthermore between those parts of the bifurcation diagram for which the corresponding precipitation condition is indeed satisfied and the parts where it is not met. In other words, the curves are demarcated into parts which are physically relevant in the context of the overall model and the parts which are not (see Fig.2). The model equations thus contain physically irrelevant equilibria too, as depicted by the thin magenta and red curves, which do not manifest in the final solution to the overall model once the corresponding precipitation regimes are taken into account. The physically valid stable velocity equilibria are described for $P = 0$, where $R < 0$, by the thick solid magenta segment; and for $P > 0$, where $R > 0$, by the thick solid red segment in Fig.2.

It is clear from these results that there is indeed a saddle-node bifurcation in each of the two precipitation regimes. However, neither of the two regimes is valid across the range of insolation R. The domains of relevance of the two solutions are thus restricted, with there being an exchange of relevance between the $P > 0$ case and $P = 0$ case at $v = 0$, where $R = 0$ (see Fig.2). These relevant segments together form the thick magenta-red bifurcation curve for the system as a whole. The exchange of relevance at $v = 0, R = 0$ is illustrated by comparing Fig.2, where the bifurcation curve for $P > 0$ case crosses the boundary between the two regimes of relevance. The thick magenta-red curve in Fig.2 clearly shows that there is no physical bifurcation in the precipitating monsoon regime of the monsoon model derived from QTCM. However, each of these curve segments are themselves part of two distinct saddle-node structures ensuing from the two contrasting assumptions about precipitation. This is clear in Figure 2 where the two saddle-node (S-N) curves appear explicitly.

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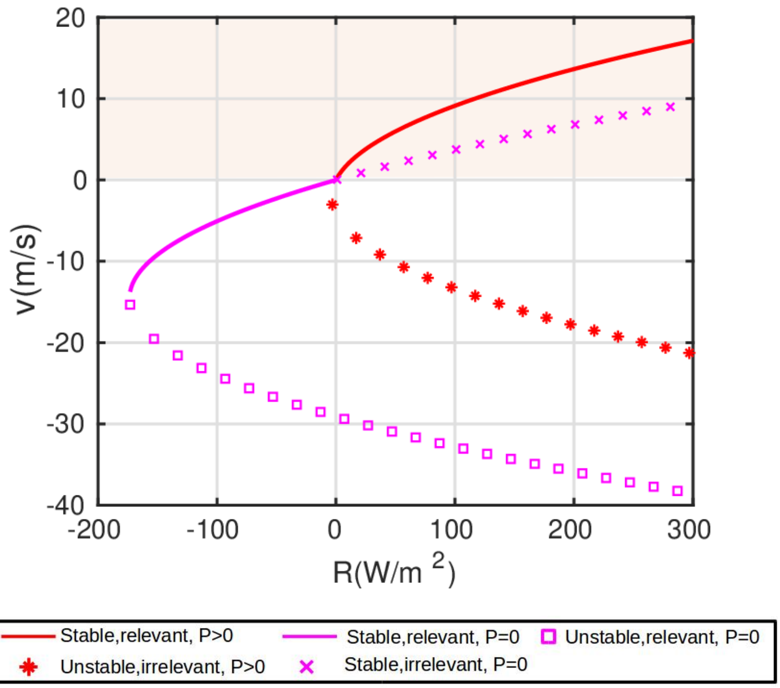

Figure 2. Standard Case of Boos & Storelvmo (2016) with stratification being present. Segments of the bifurcation curves are shaded corresponding to their stability and physical relevance. The thicker solid curves indicate the composite stable, relevant solution, which is seen to emanate from a pair of saddle-node curves. The shaded region corresponds to the portion of $v_{1s} - R$ space for which P is indeed positive, or in other words, $P > 0$ is physically relevant. Likewise, in the unshaded portion, $P = 0$, *i.e.*, $(q_{1L} - T_{1L}) \leq 0$ holds true. This illustrates that the $P > 0$ bifurcation curve (in red) spans both the regions, and must be further interpreted in light of relevance shown here.

In summary, although the final solution comprising the relevant segments itself does not contain any S-N bifurcation, the present discussion shows that this result arises from a complex of two S-N bifurcation curves that exchange their physical relevance where precipitation is zero for both the cases. The individual members of the complex do possess a S-N bifurcation and thus the figures show that equilibrium solutions are not guaranteed for all negative values of R. Since the corresponding model is applied only in the relevant regime the overall bifurcation diagram, which is the composite of the relevant portions of the two bifurcation curves, contains a single bifurcation point for large negative values of R where the v_{1s} equilibria for $P = 0$ case vanish. Furthermore, the physically important lesson is that both bifurcation points correspond to non-positive values of v_{1s} (oceanward winds in the boundary layer) and hence zero precipitation, and there is no bifurcation point with positive values of v_{1s}. As a result, the composite solution for precipitation seems to be bifurcation-free, despite the presence of a bifurcation point for large negative value of R.

The role of dry thermal stratification in the previous result becomes clear once this effect is suppressed (Fig. 3). Without dry thermal stratification, Boos and Storelvmo (2016a) reported the presence of S-N bifurcations in the model. In this study

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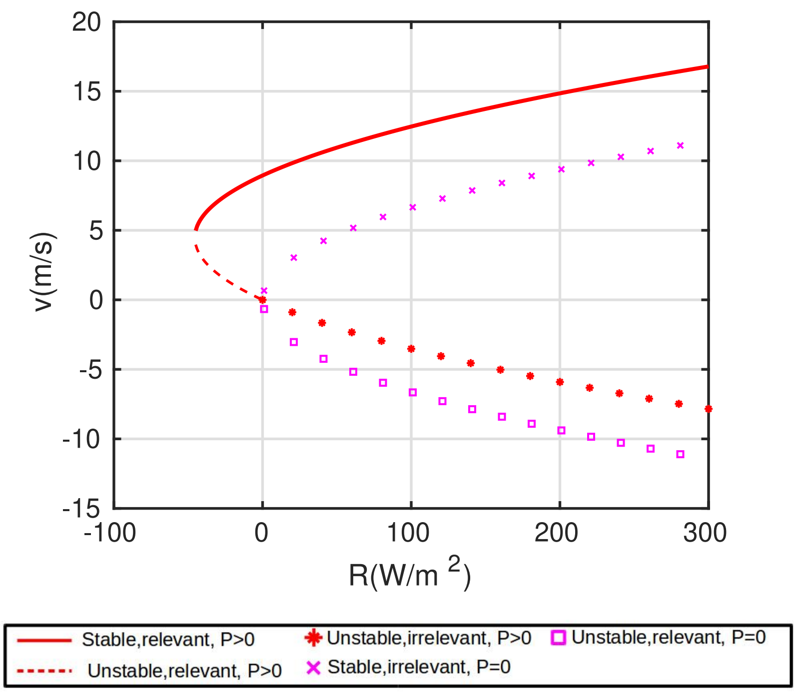

Figure 3. Standard Case of Boos & Storelvmo (2016) with stratification being suppressed.Note that the saddle-node curves persist, but they have drifted across parameter space and the composite solution, indicated by thicker solid lines manifests abrupt change.

too, without dry thermal stratification the precipitation (Fig. 4) as well as meridional velocity do show S-N bifurcation in the relevant regimes, with the resulting transition being abrupt . Notably, the underlying dynamics even here is very similar to the previous case (Fig. 2) , except for the shift in the bifurcation points of the model owing to omission of stratification. This results in an important change in the overall form of the final, stable and relevant curve. In other words, the key difference in mathematical structure of these systems is in the bifurcation points, with equilibria at the bifurcation point having $v > 0$, giving rise to abrupt onset of landward wind and consequently nonzero precipitation as the column energy source is increased. Thus, in the absence of stratification, there is a minimum wind-speed below which the monsoon circulation cannot be stable.

Figure 3 also shows that the two S-N curves intersect at $R = 0, v = 0$, where they exchange relevance.In this figure, these curves do not appear to meet elsewhere. Indeed, it turns out that in all the cases the two S-N curves appear to meet only at $R = 0$, as in the case with stratification. Quite evidently, the intersection at $R = 0$ follows from the fact that the equilibrium equations are then homogeneous and thus $v = 0$ is a common solution to both the $P > 0$ and $P = 0$ cases.

In summary, these results demonstrate that the underlying complex of S-N curves differs only quantitatively when stratification is suppressed, but this quantitative change makes all the difference to the nonlinear dynamics when the relevance of

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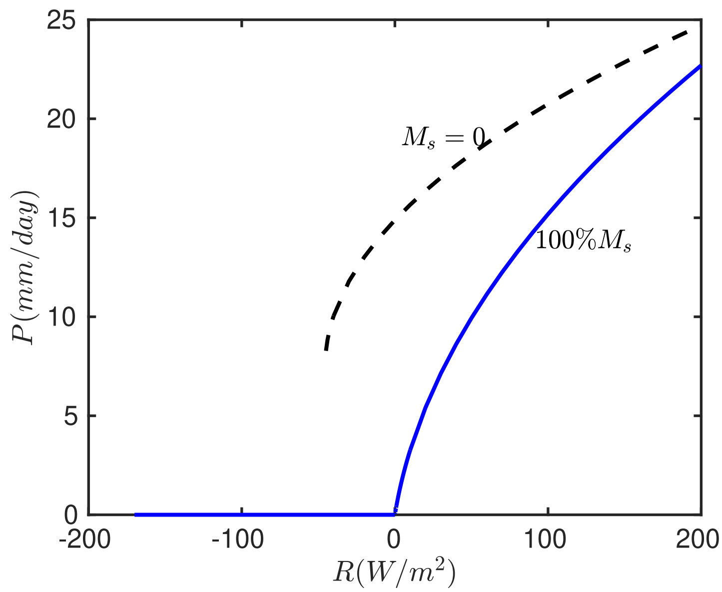

Figure 4. Precipitation corresponding to the stable, relevant equilibria in Fig.2 and Fig. 3. The results agree with those of Boos and Storelvmo (2016a), with abrupt onset of non-zero precipitation occurring for the case of suppressed stratification ($M_s = 0$, black , dashed curve).

solutions is considered. Consideration of relevance or otherwise of portions of S-N curves explains both the scenarios of the model, with and without physically relevant bifurcation being present, as stratification is either suppressed or maintained.

3.2 Effect of Stratification on Saddle-Node complex

To further understand the effects of changes in stratification on steady-states of the model, the dry thermal stratification M_s is decreased from its standard value in Boos and Storelvmo (2016a) to 0. Figure 5 shows the gradual change in the S-N curves and their interaction as M_s is varied. It is seen that, as this stratification parameter is reduced to 20% of its original value, the S-N curve corresponding to the $P > 0$ case has a bifurcation occurring for positive values of precipitation and consequently the abrupt onset phenomenon that was found earlier for the case of $M_s = 0$ has appeared. Note that such a scenario occurs only for the $P > 0$ solution. At higher stratification such as $60\%M_s$, such a bifurcation already begins to appear. Additionally, since the two solution curves exchange their physical relevance at the origin, two consequences immediately follow. First, a bifurcation is present in the overall solution whenever the bifurcation point of the $P > 0$ solution occurs at a value of v_{1s} for which indeed $P > 0$ - in other words, the bifurcation point is physically relevant. This is in contrast to the results in Fig.2 with $100\%M_s$, where the bifurcation points of both $P > 0$ and $P = 0$ are in the negative v_{1s} region. Second, owing to the manner in which these solution curves shift as the stratification is lowered, the non-monsoonal branch corresponding to $P = 0$ exists for only a short range of negative R as stratification is lowered. Both parameter values, $20\%M_s$ and $60\%M_s$, display a small regime of multistability close to $R = 0$.

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This is consistent with the result of Seshadri (2017) that lowered stratification diminishes the stabilizing influence during the monsoonal circulations, thereby making a bifurcation more likely. Thus, as the stratification is lowered further, there occurs a concomitant increase in the minimum wind-speed below which the monsoon steady state cannot exist (Figure 5). At the same time, lowered stratification gives a stronger circulation and greater precipitation for the same column energy input, as the atmospheric stabilizing effect diminishes. Overall, the monsoonal regime acquires conditions for more rainfall but only above some threshold, with there being an abrupt onset, as the stratification is lowered.

The non-monsoonal branches with $P < 0$ shift to the right as stratification diminishes substantially. Owing to their crossing through the origin they also shift upwards. Weaker stratification leads to weaker steady-state winds in this regime. Furthermore this solution branch exists over a narrower regime, with the bifurcation occurring for less negative values of the column energy source.

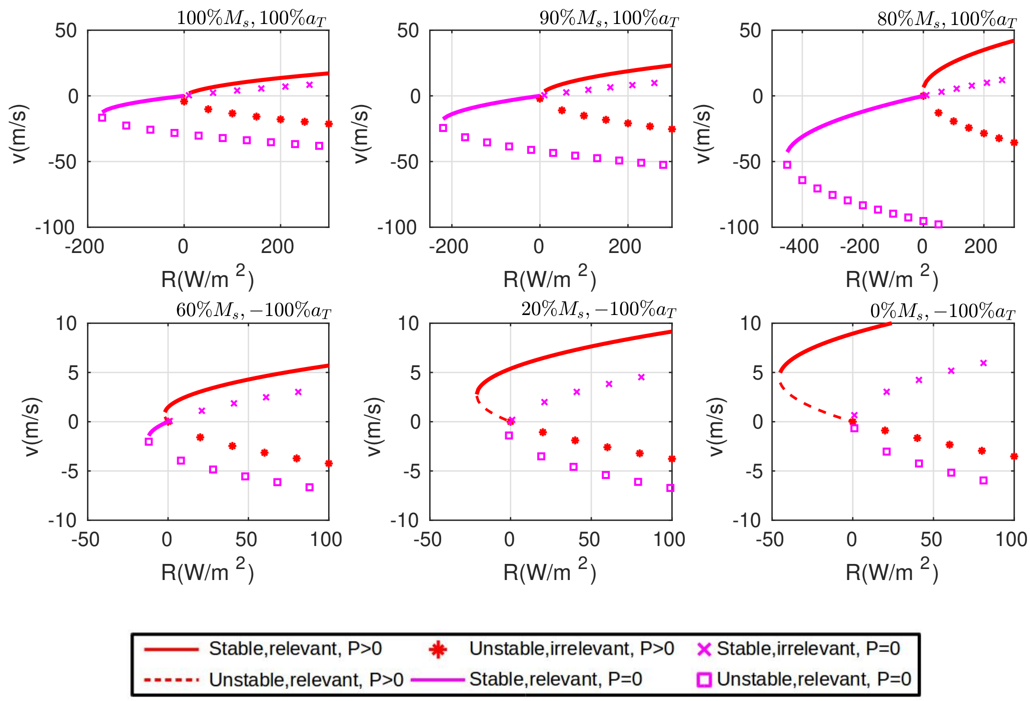

Figure 5. Standard Case of Boos & Storelvmo (2016) with stratification being present. Effect of reducing Ms from its standard value to zero shows the gradual change in dynamics, and appearance of bifurcation as the stratification is lowered. A note regarding suitable choice of a_T is given in Supplementary Information.

Figure 6 shows the effect of increasing the stratification parameter beyond $100\% M_s$, *i.e.*, the value used in the standard case. An increase of M_s much beyond the standard value seems to cause only quantitative changes in the steady states, with no qualitative changes in the bifurcation diagram. In this case, the S-N bifurcation points for both the regimes with $P > 0$

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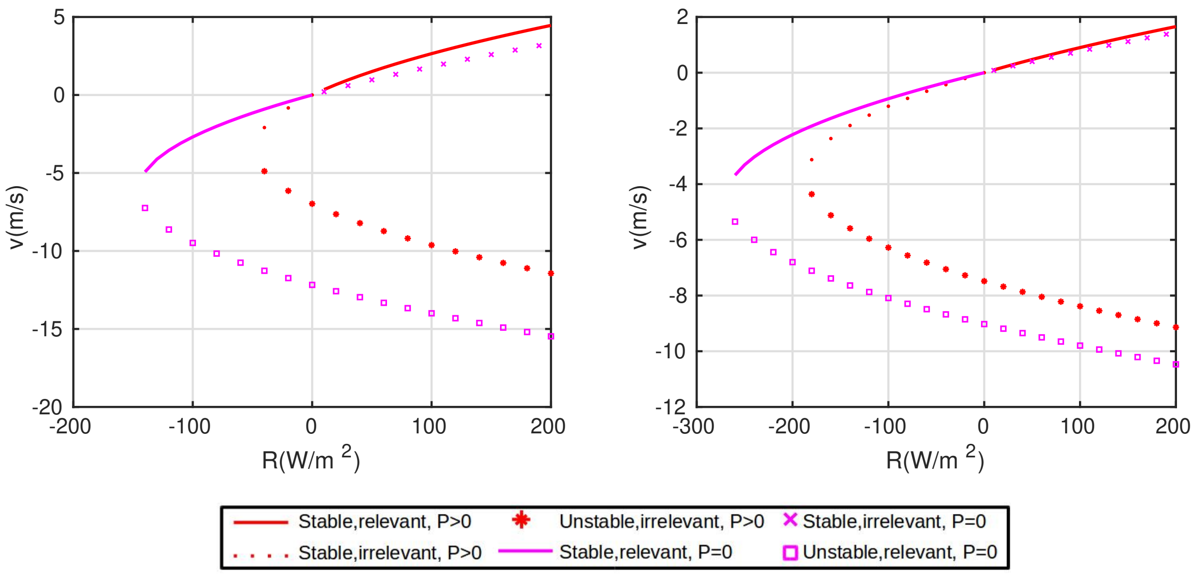

Figure 6. Standard Case of Boos & Storelvmo (2016) with stratification being present. An increase of Ms much beyond the standard value of Ms seems to cause only quantitative change but no qualitative effect on dynamics.

and $P = 0$ occur for more negative values of the column energy source as M_s is increased. This is in contrast to the case of decreasing M_s described above, where the two bifurcation curves drifted in opposite directions. In the monsoonal regime, stronger stratification diminishes precipitation. In the non-monsoonal regime, a stronger stratification imports more energy into the control volume, thereby stabilizing this solution (where the temperature gradient is reversed) for larger negative values of R.

3.3 Unfolding of Pitchfork bifurcation

Although there is a change in qualitative behavior, with physically relevant bifurcations emerging in the model as the stratification is decreased, the overall saddle-node complex structure is itself persistent to the parameter changes considered above. This explains the dynamics observed earlier in models derived from QTCM when the effect of stratification is strong (Boos and Storelvmo, 2016a). The $P = 0$ case always yields a quadratic equilibrium equation and can be expected to exhibit saddle-node structures always. However, the appearance of these saddle-node structures themselves for the cubic equation describing the $P > 0$ regime merits attention.

In special cases of the parameters, this cubic equation reduces to a quadratic. This occurs, for example, where the parameter describing the effects on the gross moist stratification (M_{qp}) cancels the moisture advection coefficient (a_q or $\langle b_1 v_1 \rangle$), ultimately yielding saddle node structures (see Eq. 4). This is because, the balance between these two parameters causes the cubic term to vanish, leading to another quadratic equilibrium equation. Thus, saddle-node structures are typical for this system

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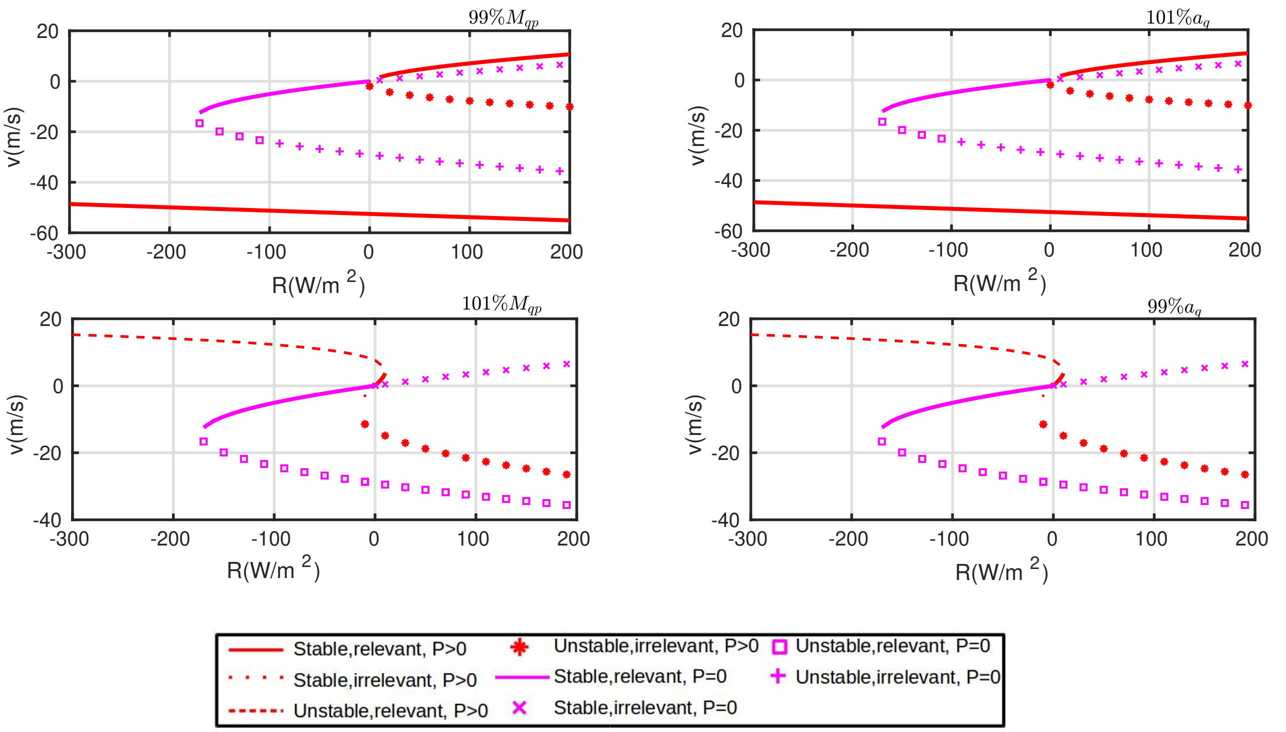

Figure 7. Standard Case of Boos & Storelvmo (2016) with stratification being present. M_{qp} is reduced from its standard value. The left panels show a supercritical and a subcritical pitchfork bifurcation as M_{qp} is slightly varied from its standard value (i.e, constraint $M_{qp} = -a_q$ is broken). The right panels show that an analogous change in a_q, keeping M_{qp} fixed at its standard value, causes the same effect. Note that the saddle node curves exist here too. However, the saddle-node curve corresponding to $P > 0$ case is now a part of the unfolding of a pitchfork bifurcation.

under these conditions. In addition, such a choice eliminates the effect of radiative forcing on the linear terms in the equilibrium equation for the velocity, so that the forcing terms, R, E, H appear together and in only one term in the equation, which allowed Boos and Storelvmo (2016a) to treat them as a single entity R.

For other cases, the cubic term would persist and the equation could be expected to exhibit other bifurcations. To study equilibria present in the more generic version of the model, where the cubic coefficient does not vanish, we modify a_q and M_{qp}. The parameter a_q describes the efficiency with which horizontal advection affects moisture, whereas M_{qp} describes the effect of moisture on the change in gross moisture stratification. When these no longer cancel, the equilibria depend on the sign of the cubic term. The results for reduction in M_{qp} are shown in Fig. 7. The left panels clearly show that the system permits pitchfork bifurcations, supercritical and subcritical respectively in the upper left and bottom left figures. This is expected, since the change in M_{qp} over and below its standard value causes the cubic term to be positive (subcritical bifurcation) and negative (supercritical) respectively.

It is notable that the quadratic curves (saddle-node) obtained for the quadratic equation analyzed earlier persist in the cubic case (red curves, $P > 0$ case) too, although as a part of unfolding of a pitchfork bifurcation. The upper left figure clearly shows that the saddle-node curves corresponding to the non-bifurcating result in Fig. 2 are almost exactly present here, with an additional, highly negative, stable equilibrium emerging due to the cubic nature. The right panels in Fig. 7 show that an increase in a_q produces the same effect as a commensurate decrease in M_{qp}. Similarly, a decrease in a_q produces the same result as a commensurate increase in M_{qp}. Not only do three equilibria result, but the plots are evidently analogous to plots for opposite changes in the value of a_q, since moisture advection and its stratification balance each other in the moisture equation. Physically, these can be interpreted as resulting from a partial cancellation between stratification and advection terms in the moisture equation, and the balance between the terms influencing the propagation of nonlinearities in the moisture field to other fields.

Notably, similar to the result in the upper left panel of Fig. 7, an unfolding of pitchfork bifurcation is present in the results of Levermann et al. (2009). However, in that study, the bifurcation was ever present owing to the absence of stratification in their model. Here, a pitchfork bifurcation occurs only when $M_{qp} \neq a_q$. In other words, when the moisture advection doesn't partially nullify the moisture stratification, the resultant system is cubic and permits pitchfork bifurcations. Further investigations into the various scenarios of unfolding while taking into account the different parameter effects are to be discussed in a subsequent paper.

4 Discussion

The results described above bring out the following:

1. A near linear response in these models arises from strong dry thermal stratification, despite bifurcations being present in the underlying saddle-node curves. Fig.2 depicts the response of the model for the standard set of parameters used in Boos and Storelvmo (2016a). The thick magenta and red curves, from the two regimes respectively, correspond to the meridional velocity equilibria of the composite model and are consistent with prior analyses of Boos and Storelvmo (2016a) for the case where the effect of stratification is included in the model. These are indeed the physically relevant and stable portions of the bifurcation curves of the set of ODEs corresponding to $P = 0$ and $P > 0$ regimes. Figure 4 shows the corresponding precipitation (solid blue curve), which also concurs with Boos and Storelvmo (2016a). The final solution comprising only the relevant segments contains no bifurcations- and therefore Boos and Storelvmo (2016a) have interpreted this as a near-linear response.

2. The saddle-node complexes discussed above are intrinsic to the low order models derived from QTCM. Therefore, they underlie the two contrasting cases of the models, where bifurcations manifest physically and where they do not. This provides the reconciliation of the absence of physical bifurcations, in the presence of large stratification, with the nonlinearities of the model. In other words, a physically relevant bifurcation depends on whether the bifurcation point for the equation set with nonzero precipitation occurs with landward winds. More generally, similar behavior can be

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envisaged for other simple atmospheric models where different equation sets are obtained under contrasting, mutually exclusive, regimes.

3. The dry thermal stratification affects each of the (linear and nonlinear) coefficients in the equilibrium equation, except the constant term (Eq.4 and Eq.5). It can be seen from these equations that a purely quadratic equation for the equilibria yields a saddle-node curve with its bifurcation point at the origin. The presence of a linear term, which generally appears in the equilibrium expressions for both the precipitating and nonprecipitating regimes, causes the point to drift in the equilibrium-parameter space. Hence, the main effect of change in stratification is to cause a drift in bifurcation curves, for both these regimes. Thus the change in stratification affects whether the bifurcation point for the precipitating regime occurs at a positive or negative value of landward winds. The former case corresponds to a physical bifurcation in the model.

4. For such systems where solutions of two regimes must be spliced, the question of physical relevance of equilibria and bifurcations might be examined considering a “critical relevance curve", distinguishing relevant and irrelevant equilibria. For the present model, this is the curve/line corresponding to $P = (q_1 - T_1)/L = 0$, dividing the parameter-variable space into region with and without precipitation (*i.e.*, $(q_1 - T_1 \leq 0)$). This naturally distinguishes the bifurcation curve for the two regimes into physically relevant and irrelevant segments, with Figure 2 illustrating this well. The overall problem is one where a saddle-node bifurcation complex, or saddle-nodes within the unfolding pitchfork, interacts with a curve separating relevant and irrelevant portions of the parameter space. This might yield, as in this case, composite bifurcation diagrams arising from simple bifurcations such as saddle-node or pitchfork, which need not directly resemble either of these canonical bifurcations, and where bistability might be an emergent property depending also on the relevance criteria. Further analysis of these systems is a promising avenue of further study. Moreover, such systems are in general non-smooth, and their transient dynamics may contain additional surprises that are beyond our present scope.

5 Conclusions

The study of whether monsoon systems exhibit bifurcations has become an important research topic, ever since low order models were first published that suggested that such phenomena are intrinsic to monsoon systems (Levermann et al., 2009). The analysis and interpretation of nonlinear dynamics using a variety of models holds lessons not only for interpretation of the paleoclimatic record, but also for understanding monsoon evolution in the context of global change. The present study is consistent with lessons from prior studies (Boos and Storelvmo, 2016a; Seshadri, 2017) that have indicated that physical bifurcations are not intrinsic to climate models and may not always be manifested by such models, because thermal stratification of the atmosphere provides an important stabilizing effect. Furthermore, when this stratification is large, such as in present-day South Asian monsoon, a bifurcation is not present.

Such a finding left open the precise effect of the nonlinear terms in these models, and this study reconciles this nonlinear dynamics with the absence of bifurcation phenomena. The resolution resides in the fact that bifurcations appear in the solu-

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tions to the model equations, however may not manifest due to their physical irrelevance. Solutions to the composite system describing the monsoon and non-monsoon regime are obtained by splicing together the respective solutions with and without precipitation. Bifurcations in the monsoonal, i.e. precipitating, regime are physically relevant only if they occur in they occur for landward winds, otherwise they do not comprise the physically relevant solution curves. This paper elaborates the behavior of the underlying complex of saddle-node bifurcations in these simple models derived form the Quasi-Equilibrium Tropical Circulation model, showing that they do not manifest if the stratification is sufficiently large. Statification is an important stablizing effect on the precipitating branch of the solution, and details of the stabilizing effects of stratification are explicable in this framework. More generally, it is seen that the saddle-node structures themselves are part of unfolding of pitchfork bifurcations. Thus, even the simplest monsoon models derived here exhibit rich dynamics, and a fuller characterization of this dynamics is an important avenue for future investigations.

Author contributions. K.K conceptualised the study, conducted the numerical experiments and analysis of the results, and wrote the original draft. A.S conceptualised the study, refined the methodology and edited the original draft.

Competing interests. The authors declare that they do not have any competing interests.

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References

Ashok, K., Guan, Z., and Yamagata, T.: Impact of the Indian Ocean dipole on the relationship between the Indian monsoon rainfall and ENSO, Geophysical research letters, 28, 4499–4502, 2001.

Bellon, G. and Sobel, A.: Poleward-propagating intraseasonal monsoon disturbances in an intermediate-complexity axisymmetric model, Journal of the atmospheric sciences, 65, 470–489, 2008a.

Bellon, G. and Sobel, A. H.: Instability of the axisymmetric monsoon flow and intraseasonal oscillation, Journal of Geophysical Research: Atmospheres, 113, 2008b.

Bellon, G. and Sobel, A. H.: Multiple equilibria of the Hadley circulation in an intermediate-complexity axisymmetric model, Journal of Climate, 23, 1760–1778, 2010.

Boos, W. R. and Storelvmo, T.: Near-linear response of mean monsoon strength to a broad range of radiative forcings, Proceedings of the National Academy of Sciences, 113, 1510–1515, https://doi.org/10.1073/pnas.1517143113, 2016a.

Boos, W. R. and Storelvmo, T.: Reply to Levermann et al.: Linear scaling for monsoons based on well-verified balance between adiabatic cooling and latent heat release, Proceedings of the National Academy of Sciences, 113, E2350–E2351, 2016b.

Bordoni, S. and Schneider, T.: Monsoons as eddy-mediated regime transitions of the tropical overturning circulation, Nature Geoscience, 1, 515–519, 2008.

Burns, S. P., Sobel, A. H., and Polvani, L. M.: Asymptotic solutions of the axisymmetric moist Hadley circulation in a model with two vertical modes, Theoretical and Computational Fluid Dynamics, 20, 443–467, 2006.

Gadgil, S. and Gadgil, S.: The Indian monsoon, GDP and agriculture, Economic and political weekly, pp. 4887–4895, 2006.

Ghil, M.: Deceptively-simple models of climatic change, Climate and Geo-sciences, pp. 211–240, 1989.

Goswami, B. N., Madhusoodanan, M., Neema, C., and Sengupta, D.: A physical mechanism for North Atlantic SST influence on the Indian summer monsoon, Geophysical Research Letters, 33, 2006.

Held, I. M.: The gap between simulation and understanding in climate modeling, Bulletin of the American Meteorological Society, 86, 1609–1614, 2005.

Jalihal, C., Srinivasan, J., and Chakraborty, A.: Modulation of Indian monsoon by water vapor and cloud feedback over the past 22,000 years, Nature communications, 10, 1–8, 2019.

Jeevanjee, N., Hassanzadeh, P., Hill, S., and Sheshadri, A.: A perspective on climate model hierarchies, Journal of Advances in Modeling Earth Systems, 9, 1760–1771, 2017.

Kucharski, F., Bracco, A., Yoo, J., and Molteni, F.: Atlantic forced component of the Indian monsoon interannual variability, Geophysical Research Letters, 35, 2008.

Levermann, A., Schewe, J., Petoukhov, V., and Held, H.: Basic mechanism for abrupt monsoon transitions, Proceedings of the National Academy of Sciences, 106, 20 572–20 577, https://doi.org/10.1073/pnas.0901414106, 2009.

Maher, P., Gerber, E. P., Medeiros, B., Merlis, T. M., Sherwood, S., Sheshadri, A., Sobel, A. H., Vallis, G. K., Voigt, A., and Zurita-Gotor, P.: Model hierarchies for understanding atmospheric circulation, Reviews of Geophysics, 57, 250–280, 2019.

Neelin, J. D. and Zeng, N.: A Quasi-Equilibrium Tropical Circulation Model—Formulation*, Journal of the Atmospheric Sciences, 57, 1741–1766, https://doi.org/10.1175/1520-0469(2000)057<1741:AQETCM>2.0.CO;2, 2000.

Palmer, T.: Stochastic weather and climate models, Nature Reviews Physics, 1, 463–471, 2019.

Polvani, L., Clement, A., Medeiros, B., Benedict, J., and Simpson, I.: When less is more: Opening the door to simpler climate models, Eos, 98, Eos, Transactions American Geophysical Union, 99, 15–16, 2017.

Pottapinjara, V., Girishkumar, M., Ravichandran, M., and Murtugudde, R.: Influence of the Atlantic zonal mode on monsoon depressions in the Bay of Bengal during boreal summer, Journal of Geophysical Research: Atmospheres, 119, 6456–6469, 2014.

Prell, W. L. and Kutzbach, J. E.: Monsoon variability over the past 150,000 years, Journal of Geophysical Research: Atmospheres, 92, 8411–8425, 1987.

Saji, N., Goswami, B. N., Vinayachandran, P., and Yamagata, T.: A dipole mode in the tropical Indian Ocean, Nature, 401, 360–363, 1999.

Schneider, S. H. and Dickinson, R. E.: Climate modeling, Reviews of Geophysics, 12, 447–493, 1974.

Schneider, T. and Bordoni, S.: Eddy-mediated regime transitions in the seasonal cycle of a Hadley circulation and implications for monsoon dynamics, Journal of the Atmospheric Sciences, 65, 915–934, 2008.

Seshadri, A. K.: Energetics and monsoon bifurcations, Climate Dynamics, 48, 561–576, 2017.

Webster, P. J., Magana, V. O., Palmer, T., Shukla, J., Tomas, R., Yanai, M., and Yasunari, T.: Monsoons: Processes, predictability, and the prospects for prediction, Journal of Geophysical Research: Oceans, 103, 14 451–14 510, 1998.

Zeng, N., Neelin, J. D., Chou, C., Wei-Bing Lin, J., and Su, H.: Climate and Variability in the First Quasi-Equilibrium Tropical Circulation Model, in: General Circulation Model Development, edited by Randall, D. A., vol. 70 of *International Geophysics*, pp. 457–488, Academic Press, https://doi.org/https://doi.org/10.1016/S0074-6142(00)80063-7, 2000.

Zhou, W. and Xie, S.-P.: A hierarchy of idealized monsoons in an intermediate GCM, Journal of Climate, 31, 9021–9036, 2018.