Dear Prof. Levermann,

I am sincerely grateful to my two anonymous reviewers. I have adopted all their comments and hope that the manuscript is now closer to their expectations. All changes are marked red.

Mikhail Verbitsky

**Response to Anonymous Referee #1,**

Dear Referee 1, I am grateful for your comments and suggestions that will help me to improve the manuscript.

**General comments:** The author claims to have proved, through the use of dimensional analysis, that the ice-climate system possesses the property of incomplete similarity, concluding from there that a certain event at orbital time-scales could have different causes.
**Answer:** Your observation is correct.

**Comment:** Several aspects are not immediately clear to me: **(1)** Definitions for many crucial terms (incomplete similarity among them) are not properly presented in the manuscript and it is, thus, very difficult to follow the logic of the paper and evaluate the potential implications.
**Answer:** Definitions of some key terms such as physical similarity, complete similarity, and incomplete similarity were not conceived by the author but have been adopted from Barenblatt (2003). The goal of such adaptation was to simplify G. I. Barenblatt's rigorous formalism and to make it a bit more explicable to a wide audience. You comment clearly testifies that this goal is yet to be achieved.
**Action:** In the new version of the paper these definitions will be edited taking into account your multiple specific comments below. **Done, new lines 46 – 56.**

**Comment: (2)** The core of the manuscript is the application of dimensional analysis to a simplified model, leading to the derivation of the adimensional parameter *V*-number. This has been done already in Verbitsky et al., 2018. I struggle to find what the added value of what is presented here is. In the current manuscript the author just modifies how the *V*-number changes in time (the *V*-number depends on several model parameters and it is, thus, possible to modify it by changing the values of those parameters in different ways). How is this a substantial knowledge contribution? Seems only an example and not enough material for a new publication.
**Answer:** This is, certainly, the most important comment, but before I respond to it (since the new version of the manuscript is not ready yet) I want you to become a bit more comfortable with some key terms. We will consider two phenomena as being *physically similar* if they are described by identical similarity parameters. For example, the air flow in a wind tunnel and a real flow are physically similar if they are both described by the same similarity parameters (like, e.g., Reynolds number) of the same value. The dimensionless time series of physically similar processes are identical. If a similarity parameter can be excluded from the description of a physical process (because, let say, it is negligibly small relative to other similarity parameters) we can talk about *complete similarity* of this physical process in this parameter: regardless of its specific value, the process doesn't depend on it. And finally, we may have *incomplete similarity*, when similarity parameters cannot be neglected even if they are small, but the number of effective parameters may still be reduced because a phenomenon depends not on absolute value of similarity parameters but on their ratios (in some power degree, i.e., conglomerate groups).

Now, we can talk about novelty. Yes, the *V*-number has been discovered experimentally (numerically) in the Verbitsky et al., 2018 (VCV18 thereafter) and has been discussed and explored extensively to demonstrate its defining role in VCV18 system dynamical properties, such as period doubling, scale invariance, *etc*. Finding incomplete similarity was not our initial target but when we figured out that VCV18 dynamics is defined by the ratio of its positive and negative feedbacks (the *V*-number) we then first become suspicious that incomplete similarity may be involved.

Indeed, the VCV18 system has 11 governing parameters, 3 of them are parameters with independent dimensions. It means that the VCV18 behavior can be fully described by 8 dimensionless similarity parameters $\pi_1 - \pi_8$ :

$$\pi_1 = \frac{\varepsilon}{a}, \pi_2 = \alpha, \pi_3 = \kappa\gamma\varepsilon T^3, \pi_4 = c\gamma\varepsilon T^3, \pi_5 = \frac{T}{\tau}, \pi_6 = \frac{\gamma T}{\beta}, \pi_7 = \frac{S_0}{\varepsilon^2 T^2}, \pi_8 = \frac{\varsigma}{\varepsilon^{1/2} T^{1/2}}$$

At the same time, we experimentally established that the period of the system response depends on smaller number of parameters, namely:

$P = T\Psi(\pi_1, \Pi_1)$
where

$$\Pi_1 = V = \left(\pi_2 + \frac{\pi_3}{\pi_4}\right)\frac{\pi_6}{\pi_5} = \frac{\gamma\tau}{\beta c}(\alpha c + \kappa)$$

It is the moment (after VCV18 has been already published) when we finally realized that we are dealing with incomplete similarity. This insight gives us much more than simply, as you say, "the *V*-number depends on several model parameters", in fact it is formed by several ***similarity parameters***. Therefore it provides us with additional powerful vision: different combinations of $\pi_i$ may produce the same *V*-number, i.e., physically unsimilar processes (formed by not identical $\pi_i$ ) may cause the same outcome.

To my knowledge this proposition is novel.

**Action:** We will include above reasoning in the paper. **Done, new lines 98-154. This analysis is novel and has not been published.**

**Comment: (3)** The author applies dimensional analysis to a 3-equations model of the ice sheets – climate system and makes some conclusions. How does the author then conclude that the real-world ice sheet- climate system also has the incomplete similarity property (whatever definition he is using)? I don't think it is possible to demonstrate, as the author claims, by the arguments shown here that the real-world system has the same a property as the simple model. There is some attempt to discuss this issue in the Conclusions, but the Abstract gives the misleading impression that a demonstration for the real-world case is provided in the paper.
**Answer:** This is definitely misinterpretation. The paper does not claim that incomplete similarity of the real-world ice-climate system has been proved. Instead, in the Conclusions (lines 146-149), the author is very explicit: "But is incomplete similarity of the global, orbital-scale, climate system real? So far, **this property has been found only in our VCV18 low-order dynamical model**, and although this model has been explicitly derived from the conservation laws, **the incomplete similarity of the ice-climate system will remain hypothetical until it is supported by empirical data**." Further, the author frames the answer to this question as a challenge for future research.

**Action:** If the Abstract contributed to such misinterpretation, its language will be revisited, otherwise no action is required. **Done, new lines 8-14.**

**Comment: (4)** Even if the real-world system would have the incomplete similarity property (again, a concept not clearly defined in the text), which are the "theoretical limits for the use of paleo climate proxy records" that the author derives? Is the author implying that proxy records have no value for understanding millennial-scale fluctuations? Even if an event can be produced by several different causes, why does this pose a theoretical limit for the use of proxy records? Which are the "far-reaching implications" that the author says might follow from the present study?
**Answer:** The goal of the paper is to demonstrate that disambiguation efforts, or in other words, attempts to attribute proxy records to a specific physical phenomenon may be fundamentally difficult if not impossible. Certainly, precise disambiguation of historical records is always a difficult task because even two *physically similar* processes having identical adimensional similarity parameters and demonstrating the same behavior may have been produced by different values of physical parameters involved, unless these parameters are physical constants or well defined. For example, suppose we have experimental measurements of an air flow made in a wind tunnel with a specific Reynolds number but the dimensions of the tunnel and air velocity data have been lost. Obviously, the same results can be produced by different tunnels having different dimensions and different flow velocities (air viscosity fortunately is the same) as long as the Reynolds number is the same. Thus, the task to attribute the results to a specific tunnel may be problematic. The situation becomes especially challenging when we deal with *incomplete similarity* because, as we discussed above, the same results may be produced by not-identical similarity parameters (physically unsimilar processes). This is the theoretical limit that we aspire to expose.
**Action:** The goal of the paper will be better articulated. **Done, new lines 30, 57-64, 146-154.**

**Comment:** Overall, I find that the manuscript does not provide clear definitions for crucial terms (for example: physical similarity, complete and incomplete similarity) and is, therefore, difficult to read in its current form. Furthermore, given that a dimensional analysis has already been applied to the same (or a very similar) model in Verbitsky et al. 2018, how the material presented here constitutes a substantial new contribution is not at all evident. The differentiation from Verbitsky et al. 2018 must be made clearer and the new contributions highlighted. In its current form, the material here presented seems to be a simple illustration or example (V-value changing by changing the different model parameters that affect it).
**Action: See responses and action items to comments (1) – (2).**

**Particular comments:** I find that the introduction section must be substantially reformulated. Clear definitions and notations must be introduced. In addition, the author should clearly state the goal of the manuscript, which is missing in the current version.
**Action: See response and action items to comments (1) - (4).**

**Lines 9-10:** This sentence is not clear enough for an abstract (and repeats the word similarity 3 times). I would reformulate it and link to the following sentence. For example: Specifically, we demonstrate that major past events could have been produced by different physical processes and, therefore, the task of disambiguation of the historical paleo-records may be fundamentally difficult, if not impossible.
**Action:** The language will be revisited. **Done, new lines 10-13.**

What is it there to disembogue? Please clarify

**Action: See response and action item to comment (4)**

**Lines 13-14:** Are you implying that glaciations are independent of orbital forcing? Is this paper implying we cannot forecast events at orbital time-scales? Any proofs?
**Answer:** There is no implication here that glaciations are independent of orbital forcing. There is no implication that we cannot forecast events at orbital time-scales. Instead, we demonstrate that orbital-timescale predictions are, possibly, more stable than we might have thought – same results may be produced by multiple scenarios as long as they are based on the same *V*-number.
**Action:** This will be clarified **Done, new lines 13 – 14, 198-201.**

**Lines 21-24:** Please cite the work of Willeit et al. (2019).Willeit, M., Ganopolski, A., Calov, R., & Brovkin, V. (2019). Mid-Pleistocene transition in glacial cycles explained by declining CO2 and regolith removal. *Science Advances*, *5*(4),eaav7337.
**Action:** Thank you. It will be done **Done.** I also included Riechers, K., Mitsui, T., Boers, N., and Ghil, M.: Orbital Insolation Variations, Intrinsic Climate Variability, and Quaternary Glaciations, Clim. Past Discuss. [preprint], https://doi.org/10.5194/cp-2021-136, in review, 2021.

**Line 34:** change super long to extremely long. Also specify what does the author understand by orbital time scales in yrs.
**Action:** It will be done **Done, new line 35**

**Line 40:** remove the word prophetic.
**Action:** It will be done. **Done.**

**Line 47:** asymptotic in which direction? What is the meaning of a parameter to disappear? Please, clarify, this is not clear.
**Action:** It will be clarified. **Done, new lines 48 - 50**

**Lines 48-49:** what are the super indices mi, ni, qi? it is not defined. What does it mean pi in between brackets? The notation is not understandable, as it is not clearly defined. What's the meaning of conglomerate group?
**Action:** Thank you. I agree that notation here is not clear. It will be corrected. **Done, new lines 51-56**

**Line 52**: discover? did you prove the real-world system has this property? Or just your system of equations?
**Action: See response to Comment 3.**

**Line 55**: what is the definition of conglomerate *V*-number?
**Action:** The definition will be provided **Done, new lines 107 - 122**

**Line 60:** You said you already did this in VCV18, why do it again here?
**Action: See response to Comment 2**

**Lines 69-72:** Please, provide some extra description of the model. It is not obvious how terms to the power of 3/4 or 1/4 appear from the fundamental equations. A brief description of the main assumptions considered in the model derivation would be useful here. Also, mention if the model has been successfully validated against paleo data, or any indication of its ability to represent the real-life system.

**Action:** Additional description will be provided **Done, new lines 76 - 93**

**Line 76:** what is epsilon? It has units of velocity
**Action:** Clarification will be provided **Done, new lines 85-86**

**Line 78:** Are you referring now to the real-world dynamical system or the one of your equations?
**Action:** It will be clarified **Done, new line 94**

**Line 88:** terrestrial ice sheet mass flux was earlier called snow precipitation rate. Please, use only one name.
**Action:** It will be done **Done, new line 109**

**Line 93:** How is it derived that V represents that? Which are the positive/negative feedbacks? Below it says: beta is intensity of negative feedbacks, gamma is intensity of positive feedbacks. What is intensity??
**Action:** Although all these have been discussed in VCV18, I will briefly describe it here for clarity **Done, new lines 110-120**

**Line 95:** what is the meaning of slow here?
**Action:** Thank you. I agree that it needs clarification. It will be done. **Done, new lines 125 - 128**

**Line 100:** define the Peclet number
**Action:** It will be done **Done, new line 113**

**Line 104:** "This feedback is applied directly to the ice sheet mass balance ($\gamma\tau\kappa$)" where is this derived?
**Action:** Although all these have been discussed in VCV18, I will briefly describe it here for clarity. **Done, new lines 110 - 120**

**Line 105:** how is the $\gamma\tau\alpha c$ coefficient derived?
**Action:** Although all these have been discussed in VCV18, I will briefly describe it here for clarity **Done, new lines 110 - 120**

**Line 106:** what's the meaning of non-idealized?
**Answer:** "Non-idealized" here means that we solve pure system (1) – (3) without any parameters being neglected
**Action:** It will be clarified. **Done, "non-idealized" removed**

**Line 107:** Please explain the meaning of "invoking a global cooling trend"
**Action:** It will be clarified **Done, new lines 164-166**

**Line 109:** define continentality
**Action:** It will be clarified **Done, new line 167**

**Line 112:** what does the *m* subscript represent?
**Answer:** "*m*" is for millennium
**Action:** No action is required. **Actually I got rid of it. New line** 170

**Figure 1:** indicate that x-axis is time (kyr). Please also show the S time-series in each case.

**Action:** Thank you, time will be explained. **Done, Fig. 1: S-time series are included as inserts and lines 322-325**

**Line 130:** The explanation of which astronomical forcing is used should be mentioned much earlier in the text.
**Action:** It will be done. **Done, new lines 170-172**

**Line 146:** "But is incomplete similarity of the global, orbital-scale, climate system real?" I think there is no answer to this question in the manuscript, therefore, you can only focus in your model characteristics.
**Action:** This has been discussed in my response to Comment 3. No additional action is required

**Response to Anonymous Referee #2,**

Dear Referee 2, I am grateful for your comments and suggestions that will help me to improve the manuscript.

**Summary:** Overall, the goal of this manuscript is to demonstrate that a reduced order model of ice sheets exhibits incomplete similarity. I will be honest that I found this study to be hard to follow. I apologize to the authors in advance, if misunderstood what they did or said. Based on the difficult I had following the approach, I might not be the right person to review this manuscript. Nonetheless, my comments are below.
**Answer:** Thank you for your efforts. I am the one who should apologize for bringing to your attention a manuscript that is not explicable on its own. Yes, the goal of the paper, as you correctly observed, is "to demonstrate that a reduced-order model of ice sheets (*and climate* – MV) exhibits incomplete similarity". But this is only part of the goal. Most importantly, I wanted to demonstrate that, because of incomplete similarity, different combinations of similarity parameters (physically unsimilar processes) may lead to the same outcome.

I will explain this in my detailed answers below.

**Comment:** The basic system of equation is presented early on. It would make the manuscript much more accessible to provide an expanded description of the model and the physical interpretation of the parameters. For example, the parameter "a" is described as a snow precipitation rate. But the snowfall rate depends on the climate. Glacial cycles are known to be drier than interglacial cycles. And does the snowfall rate also include the melt rate? Or is that specified separately? Clearly, the melt rate has to depend on climate doesn't it? And then there are a host of "sensitivity coefficients". What do these physically represent and how would I measure them?
**Answer:** The model used in this study has been extensively described in Verbitsky et al, 2018 (VCV18 thereafter), and all questions you are raising above have been addressed there. I agree with you though that, for convenience of our readers, additional model description would be helpful.
**Action:** Additional model description will be provided. **Done, new lines 76-93**

**Comment:** My next question arises from the assertion of incomplete similarity and description of what this means. Now I am vaguely familiar with similarity and incomplete similarity. The author's first assertion is that the period of the system only depends on two nondimensional numbers. Here it would be helpful to provide estimates of the physical magnitudes of each of the parameters based on whatever observations are available and to provide a physical interpretation for the "V-number" and why this controls the period. But I think my biggest question is I cannot follow the connection between the period doubling and incomplete similarity. The typical definition of complete similarity is usually that the similarity function becomes independent of some non-dimensional group in the limit that the non-dimensional number tends to zero or infinity. By contrast, the definition of incomplete similarity is that the similarity function does not become independent of the non-dimensional group as the group tends to zero or infinity. Instead, you end up with a scaling law where the scaling function becomes proportional to the non-dimensional group to some power. As the authors note, it is not usually possible to determine the scaling power by dimensional analysis alone. In the exposition (line 94), neither of the parameters tend to zero or infinity. The one parameter is 1 and the other is 0.75, neither of which can be considered large or small compared to one. So then the question: what does this have to do with incomplete similarity? When I have done calculations to determine incomplete similarity the goal has usually been to determine the scaling exponent, but I am uncertain if the authors even tried to find the scaling exponent. I apologize to the authors if I misunderstood the analysis or their approach. Maybe I'm coming at it from the wrong direction.

**Answer:** Your understanding of complete and incomplete similarity is indeed correct, but, yes, in this study we approached incomplete similarity from a different direction. Finding of incomplete similarity was not our initial goal. We have been motivated to find physics responsible for period doubling bifurcation, and when we figured out that it is defined by the ratio of positive and negative feedbacks in the system (the *V*-number), we then first become suspicious that incomplete similarity may be involved.

Indeed, the VCV18 system has 11 governing parameters, 3 of them are parameters with independent dimensions. It means that the VCV18 behavior can be fully described by 8 dimensionless similarity parameters $\pi_1 - \pi_8$ :

$$\pi_1 = \frac{\varepsilon}{a}, \pi_2 = \alpha, \pi_3 = \kappa\gamma\varepsilon T^3, \pi_4 = c\gamma\varepsilon T^3, \pi_5 = \frac{T}{\tau}, \pi_6 = \frac{\gamma T}{\beta}, \pi_7 = \frac{S_0}{\varepsilon^2 T^2}, \pi_8 = \frac{\varsigma}{\varepsilon^{1/2} T^{1/2}}$$

At the same time, we experimentally established that the period of the system response depends on smaller number of parameters, namely:

$$P = T\Psi(\pi_1, \Pi_1)$$

where

$$\Pi_1 = V = \left(\pi_2 + \frac{\pi_3}{\pi_4}\right)\frac{\pi_6}{\pi_5} = \frac{\gamma\tau}{\beta c}(\alpha c + \kappa)$$

It is the moment when we finally realized that we are dealing with incomplete similarity. This insight provides us with additional powerful vision: different combinations of $\pi_i$ may produce the same *V*-number, i.e., physically unsimilar processes (formed by not identical $\pi_i$ ) may cause the same outcome. To my knowledge this proposition is novel.

**Action:** We will include above reasoning in the paper. **Done, new lines 98-154**

**Comment:** I also did not understand the figures provided. The x-axis and colorer aren't labeled and the y-axis doesn't have units. What are we supposed to see here?

**Answer:** The horizontal axis is time (kyr before present), the vertical axis is the period of the system response (kyr), the color scale shows the continuous Morlet wavelet amplitude.

**Action:** The figure legend will be updated. **Done, new lines 319-325**

**Comment:** There is another subtle issue with the analysis which is that it is always difficult to determine if the behavior of a mathematical model is a feature of the simplifications of the mathematical model or is common to the more nuanced physics that is more representative of the physical system. Here it is unclear if the authors are claiming that their simplified model obeys incomplete similarity or if the general ice-climate system obeys incomplete similarity.
**Answer:** In this regard, the author is very explicit - see Conclusions (lines 146-149): "But is incomplete similarity of the global, orbital-scale, climate system real? So far, **this property has been found only in our VCV18 low-order dynamical model**, and although this model has been explicitly derived from the conservation laws, the **incomplete similarity of the ice-climate system will remain hypothetical until it is supported by empirical data**." Further, the author frames the answer to this question as a challenge for future research.
**Action:** If the Abstract contributed to such misinterpretation, its language will be revisited, otherwise no action is required. **Done, new lines 8-14**

# Inarticulate past: Incomplete similarity of the ice-climate system and its implications for paleo-records attribution

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**Abstract.** Reconstruction and explanation of past climate evolution using proxy records is the essence of paleoclimatology. In this study, we use dimensional analysis of a dynamical model on orbital time-scales to recognize theoretical limits of such forensic inquiries. Specifically, we demonstrate that incomplete similarity does not imply physical similarity and therefore major past events could have been produced by different physical processes making the task of paleo-records attribution to a particular phenomenon to be fundamentally difficult, if not impossible. It also means that any future scenario may not have a unique cause and, in this sense, the orbital time-scale future may be to some extent less sensitive to specific terrestrial circumstances.

## Introduction

Interpretation of most prominent events of climate history such as the middle-Pleistocene transition (Ruddiman et al., 1986, Lisiecki and Raymo, 2005, Clark et al., 2021) has been an inspiration for several generations of climate modelers (see for a review Saltzman, 2002, Clark, et al., 2006, Tziperman et al., 2006, Crucifix, 2013, Mitsui and Aihara, 2014, Paillard, 2015, Ashwin and Ditlevsen, 2015, Verbitsky et al, 2018, Willeit et al., 2019, Riechers et al, 2021). While specific physical mechanisms invoked to explain changing glacial rhythmicity vary, they all include slow changes of ocean-atmosphere governing parameters (e.g., Saltzman and Verbitsky, 1993; Raymo, 1997; Paillard and Parrenin, 2004) or glaciation parameters (Clark and Pollard, 1998). On a more general level, all these theories in fact assume slow changes in the intensities of positive (such as, for example, long-term variations in carbon dioxide concentration, e.g., Saltzman and Verbitsky, 1993) or negative (for example, regolith erosion, Clark and Pollard, 1998, or vertical temperature advection in ice sheets, Verbitsky and Crucifix, 2021) system feedbacks. Though all physical phenomena invoked are, indeed, real and may be plausible, the following question still remains unanswered: Is it possible to disambiguate the past and elevate a single "correct" theory? Answering this question is the goal of our study.

Indeed, this is the classical attribution challenge that has been successfully addressed in the context of another well-known problem of geophysics: the causality of the observed global warming. For this purpose, the most comprehensive space-resolving models have been employed to reproduce observed time-series under different conditions and to prove (or discredit) a candidate physical phenomenon (e.g., Stocker, 2014). Certainly, these models cannot be employed on extremely long orbital time-scales (10 – 100 kyr) due to computational constrains. In search for an alternative, we turn here to dimensional analysis. Historically, dimensional analysis and concepts of similarity have been used for studying physical phenomena, complementing even the most sophisticated computational tools and providing physical insight in situations where physical interpretation of the higher-complexity modeling results may be difficult. Here, on orbital timescales, when we retreat from physics-abundant space-resolving models to more conceptual dynamical models, dimensional analysis may be promoted from a supporting to a more prominent~~, prophetic,~~ role.

Several key terms need to be introduced before we outline the structure of our paper. We will be using the definitions of physical similarity and complete and incomplete similarity as they have been articulated by G. I. Barenblatt (2003). Suppose we have a physical phenomenon that is governed by *n* physical parameters, *k* parameters of which are parameters with independent dimensions. Then, according to the π-theorem (Buckingham, 1914), the phenomenon can be described by *n-k* adimensional similarity parameters $\pi_1, \pi_2, \ldots \pi_i \ldots, \pi_{n-k}$. We will consider two phenomena as being *physically similar* if they are described by identical similarity parameters $\pi_1, \pi_2, \ldots \pi_i \ldots, \pi_{n-k}$. The dimensionless time series of physically similar processes are also identical. If a similarity parameter $\pi_i$ can be excluded from the description of a physical process (a phenomenon becomes independent of it in the limit that $\pi_i$ tends to zero or infinity) we can talk about *complete similarity* of this physical process in this parameter: regardless of its specific value, the process does not depend on it. And, finally, we may have *incomplete similarity* when none of similarity parameters $\pi_1, \pi_2, \ldots \pi_i \ldots, \pi_{n-k}$ can be neglected even if they are too small (or too big), but the number of effective parameters may still be reduced because a phenomenon depends not on absolute value of similarity parameters but on their products in some power degree (i.e., conglomerate groups): $\Pi_j = \left(\pi_1^{\alpha_j}\right)\left(\pi_2^{\beta_j}\right)\dots\left(\pi_i^{\lambda_j}\right)\dots\left(\pi_{n-k}^{\chi_j}\right)$ $(j = 1,2,\dots,l; l < n-k)$. Here $\alpha_j, \beta_j, \dots, \lambda_j, \dots, \chi_j$ are power degrees of $\pi_1, \pi_2, \dots \pi_i \dots, \pi_{n-k}$ involved into $\Pi_j$ formulation.

We are now ready to proceed with the structure of our paper: (a) first, we will introduce our dynamical model and describe major physical processes involved; (b) using dimensional analysis, we will define 8 similarity parameters $\pi_1 - \pi_8$ that completely define model's behavior; (c) while these adimensional similarity parameters $\pi_1 - \pi_8$ will be determined using simple rules of dimensional analysis, there are no specific algorithms that can help us in finding their effective conglomerate groups $\Pi_j$, if they indeed exist. Therefore, we will articulate such conglomerate groups based on observed system behavior; (d) we will then discuss implications of our findings for the attribution challenge and illustrate our reasoning with a numerical experiment; (e) we will conclude our study with some thoughts relating our results to the real-world climate system.

**Method**

For our experiments we employ the Verbitsky et al (2018), VCV18 thereafter, dynamical model of the ice-climate system. It has been derived from the scaled mass- and heat-balance equations of the non-Newtonian ice flow, i.e., equations (1) and (2), correspondingly, and combined with an energy-balance equation of the global climate temperature (3):

$$\frac{dS}{dt} = \frac{4}{5}\zeta^{-1}S^{3/4}(a - \varepsilon F_S - \kappa\omega - c\theta) \quad (1)$$

$$\frac{d\theta}{dt} = \zeta^{-1}S^{-1/4}(a - \varepsilon F_S - \kappa\omega)\{\alpha\omega + \beta[S - S_0] - \theta\} \quad (2)$$

$$\frac{d\omega}{dt} = -\gamma[S - S_0] - \frac{\omega}{\tau} \quad (3)$$

Here, $S$ ($m^2$) is the area of glaciation, $\theta$ ($^oC$) is the basal ice-sheet temperature, and $\omega$ ($^oC$) is the global temperature of the ocean-atmosphere (rest of the climate) system. In deriving equations (1) and (2) we considered ice sheets in the thin-boundary-layer approximation such that their inertial forces are negligible relative to stress gradients, and motion equations with very high accuracy can be written in a quasi-static form. For such approximation, a characteristic ice thickness $H$ is connected to ice area $S$ as $H = \zeta S^{1/4}$ where $\zeta$ ($m^{1/2}$) is a profile factor assumed to be constant (Verbitsky and Chalikov, 1986, VCV18). Further, equation (1) represents global ice balance $\frac{d(HS)}{dt} = AS$, where, again, $H = \zeta S^{1/4}$ and $A = a - \varepsilon F_S - \kappa\omega - c\theta$ is the surface mass influx. Equation (2) describes vertical ice temperature advection with a time scale $H/(a - \varepsilon F_S - \kappa\omega)$, and equation (3) is the global energy-balance equation. The parameter $a$ ($m\ s^{-1}$) is the snow precipitation rate; $F_S$ is normalized external forcing, specifically, mid-July insolation at 65°N (Berger and Loutre, 1991) of the amplitude $\varepsilon$ ($m\ s^{-1}$) such that $\varepsilon F_S$ describes ice ablation rate due to astronomical forcing; $\kappa\omega$ is the ice ablation rate representing the cumulative effect of the global climate on ice-sheet mass balance; $c\theta$ represents ice discharge due to ice-sheet basal sliding; $\alpha\omega$ is basal temperature response to global climate temperature change, $\beta[S - S_0]$ is basal temperature reaction to the changes of ice geometry; $-\gamma[S - S_0]$ describes global temperature response to ice geometry changes (e.g., albedo); $\kappa$ ($m\ s^{-1}\ {}^oC^{-1}$), $c$ ($m\ s^{-1}\ {}^oC^{-1}$), $\alpha$ (adimensional), $\beta$ ($^oC\ m^{-2}$) and $\gamma$ ($^oC\ m^{-2}\ s^{-1}$) are sensitivity coefficients; $S_0$ ($m^2$) is a reference glaciation area; and $\tau$ (s) is the timescale for $\omega$. When orbitally forced, the model reproduced events of the last million years reasonably well, except for the interglacial of 400 kyr ago (marine isotopic stage 11). The timing of all other interglacials coincides with Past Interglacial Working Group of PAGES (2016) data (VCV18).

We will now focus on the most remarkable feature of the historical records - a period $P$ of climate response to the astronomical forcing. Indeed, it is the change of the climate variability from the predominant period $P$ = 40 kyr to the main periods of $P$ = 80-120 kyr that makes the middle-Pleistocene transition so extraordinary. Though the amplitude increase was considered, until recently, to be a necessary attribute of this transition, its presence in the paleo-records is now questioned (Clark et al, 2021). We begin with the dimensional analysis of the VCV18 system (1) – (3). Indeed, it has 11 governing parameters (including the amplitude $\varepsilon$ and the period $T$ of the external forcing). If we choose $\varepsilon$, $T$ and $\gamma$ to be parameters with independent dimensions, then in accordance with $\pi$-theorem a period of the system response can be fully described by 8 dimensionless similarity parameters $\pi_1 - \pi_8$ :

$\pi_1 = \frac{\varepsilon}{a}, \pi_2 = \alpha, \pi_3 = \kappa\gamma\varepsilon T^3, \pi_4 = c\gamma\varepsilon T^3, \pi_5 = \frac{T}{\tau}, \pi_6 = \frac{\gamma T}{\beta}, \pi_7 = \frac{S_0}{\varepsilon^2 T^2}, \pi_8 = \frac{\varsigma}{\varepsilon^{1/2} T^{1/2}}$, and

$$P = T\Psi(\pi_1, \pi_2, \ldots, \pi_8) \quad (4)$$

At the same time, we observed earlier (Verbitsky and Crucifix, 2020) that the period of the system (1) – (3) response to the obliquity forcing of period *T* is mostly governed by two dimensionless parameters: by the ratio of the astronomical forcing amplitude to terrestrial ice sheet snow precipitation rate, $\varepsilon/a$, and by the adimensional *V*-number. The physical meaning of the *V*-number in the orbital domain becomes most evident if we take a closer look into the structure of positive and negative feedbacks as they appear in the system (1) – (3). The time-dependent negative feedback is proportional to the ice sheet area size as $\beta(S - S_0)$. The coefficient $\beta$ is defined by thermodynamical properties of an ice sheet, most importantly by the Peclet number, $Pe = \hat{A}H/k$, $\hat{A}$ is a characteristic mass influx, i.e., accumulation minus ablation and *k* is ice temperature diffusivity (VCV18, Verbitsky and Crucifix, 2021). This negative feedback acts on ice-sheet mass balance with a vertical-advection time delay and is amplified by a sensitivity coefficient *c* that reflects the intensity of basal sliding. The time-dependent positive feedback is global temperature $\omega$. In the orbital domain, $\tau \ll T$ ($\pi_5 \gg 1$), $\omega$ is approximately proportional to $-\gamma\tau(S - S_0)$. The global temperature acts on the ice-sheet mass balance "instantly" as $\kappa\omega$ and with the vertical-advection time-delay as a component of basal temperature conditions, $\alpha\omega c$. Thus, the *V*-number is emerging in the orbital domain as a ratio of amplitudes of time-dependent positive and negative feedbacks.

$$V = \frac{\gamma\tau}{\beta c}(\alpha c + \kappa) \quad (5)$$

Specifically, when $V \sim 0.75$ and $\varepsilon/a \sim 1$, the system exhibits the obliquity-period doubling. When the positive feedback and the obliquity forcing are less articulated, the system responds with the 40-kyr period. Thus, slow changes of the *V*-number (for example, from $V = 0.5$ at $t = 3{,}000$ kyr ago to $V = 0.75$ at $t = 0$) and of the $\varepsilon/a$ ratio (for example, from $\varepsilon/a = 0.3$ to $\varepsilon/a = 1.7$ over the same time span) produce a change in the ice-climate behavior similar to the middle-Pleistocene transition.
We now notice that the *V*-number can be presented in terms of similarity parameters $\pi_1 - \pi_8$, specifically:

$$V = \frac{\gamma\tau}{\beta c}(\alpha c + \kappa) = \left(\pi_2 + \frac{\pi_3}{\pi_4}\right)\frac{\pi_6}{\pi_5} \quad (6)$$

We also experimentally established that the period-doubling sustains ($\Psi = 2$) if, under fixed $\varepsilon/a$ and *V*, the period of the external forcing changes from let say $T = 35$ kyr to $T = 50$ kyr. It can only happen if in this domain similarity parameters $\pi_7$ and $\pi_8$ make another conglomerate group that does not depend on *T*, specifically $\frac{\pi_8^4}{\pi_7}$.
Thus, equation (4) can be written as:

$$P = T\Psi\left(\pi_1, \frac{\pi_2\pi_6}{\pi_5}, \frac{\pi_3\pi_6}{\pi_4\pi_5}, \frac{\pi_8^4}{\pi_7}\right), \quad (7)$$

that is the pure case of incomplete similarity as we defined it above. Finally we may notice that $\frac{\pi_8^4}{\pi_7} = \frac{H^4}{S_0^2} \ll 1$ for all large ice sheets (thin-boundary-layer approximation). If we set it to be constant and apply generalized $\pi$-theorem (Sonin, 2004) we can re-write equation (7) in a more simple form as

$$P = T\Psi\left(\frac{\varepsilon}{a}, V\right) \quad (8)$$

Recognition of incomplete similarity is important because it provides us with a powerful insight: different combinations of similarity parameters $\pi_i$ may produce the same *V*-number, i.e., *physically unsimilar processes* (formed by not identical $\pi_i$) *may cause the same outcome.* This observation is critical for our attribution challenge. Certainly, precise disambiguation of historical records is always a difficult task because even two physically similar processes having identical adimensional similarity parameters and demonstrating the same behavior may have been produced by different values of physical parameters involved, unless these parameters are physical constants or well defined. The situation becomes especially challenging when we deal with incomplete similarity because, as we just stated, the same results may be produced by not-identical similarity parameters (physically unsimilar processes). This is the theoretical limit that we aspire to expose.

We will now apply our findings to the middle-Pleistocene transition. Since the physical interpretation of the governing parameters incorporated in the conglomerate *V*-number is very straightforward, we may observe a similar (in terms of the period-*P* bifurcation) system response to changes of a completely different physical nature. For example, parameter $\beta$, as we have discussed above, defines intensity of the negative feedback and is formed as a result of interplay between vertical ice advection, internal friction, and geothermal heat flux (VCV18). Increased Peclet number of the growing ice sheet diminishes the role of the geothermal heat flux and may reduce parameter $\beta$ thus increasing the *V*-number. The same period-*P* bifurcation can also be caused, for example, by slow changes in the parameter $\gamma$ that defines the intensity of the positive feedback and incorporates effects of the albedo change or other atmospheric feedbacks. We solve equations (1) – (3) for two cases we have just described. In both cases we invoke a global cooling trend. In our first experiment (Fig. 1a), this trend is translated into reduction of $\beta$, i.e., weakening of the ice sheet negative feedback, and corresponding increase of the *V*-number from $V = 0.5$ to $V = 0.75$. We assume here that in growing ice sheets the role of the geothermal heat flux is diminished. The increased continentality of the climate (reduced intensity of the snowfall during colder climate) is accounted by the $\varepsilon/a$ ratio increase from $\varepsilon/a = 0.3$ to $\varepsilon/a = 1.7$. In the second experiment (Fig. 1b), the *V*-number also evolves from $V = 0.5$ to $V = 0.75$, but this time it is achieved by increased intensity of the positive feedback ($\gamma$). The millennial forcing is added to $\varepsilon F_S$ as a single sinusoid of 5 kyr period and doubled ($2\varepsilon$) amplitude. In both experiments, we used mid-July insolation at 65°N (Berger and Loutre, 1991) for the last 3 million years as an astronomical forcing. It is important to note that in the first experiment (changing $a$ and $\beta$) only similarity parameters $\pi_1$ and $\pi_6$ are being changed, but in the second experiment (changing $a$ and $\gamma$) the same changes of the *V*-number are caused by changing $\pi_1, \pi_3, \pi_4, \pi_6$. It means that the processes involved in these two experiments are not physically similar. Though the time-series produced in these two cases are obviously non-identical (see Fig. 1 inserts), we can observe that different physical phenomena may produce the same changes in the conglomerate *V*-number and the same large-scale effect, i.e., the period-doubling bifurcation at about 1 Myr ago.

We do not attempt here to fully reproduce paleo-records such as the Lisiecki and Raymo (2005) or Clark et al. (2021), and a discussion of whether a period doubling should be accompanied by the amplitude increase is outside of the current paper's scope. We will just remark that the amplitude of the system response is the function of not just the period *P* but also of the $\varepsilon/a$ ratio (Verbitsky and Crucifix, 2020) and, for example, less articulated continentality of colder climates may explain diminished amplitude contrasts as it has been recently advocated by Clark et al (2021).

Indeed, as we have already indicated, we used mid-July insolation at 65°N (Berger and Loutre, 1991) for the last 3 million years as an astronomical forcing. Apart from that, these examples may also serve as an illustration of some future scenarios of the climate system behavior under post-industrial atmospheric carbon dioxide concentration reduction as implied by Ridgwell and Hargreaves (2007). Again, regardless of the physical nature of the underlying dynamical system, it exhibits 40-kyr rhythmicity of the first 1.5 million years of its evolution and consequent obliquity-period doubling. This probable renaissance of ice-ages is different from the one envisioned by Talento and Ganapolski (2021) which is based on the model tuned to the late Pleistocene (last 800 kyr) ice-volume data and thus postulates only 100-kyr-period variability for the future.

## Conclusions

The idea of the current presentation is simple but its implication may be important: If ice-climate system has a property of incomplete similarity, then we may be limited in our ability to disambiguate historical records and different physical processes may produce same future scenarios. The latter is intriguing because since B. Saltzman (1962) and E. Lorenz (1963) had discovered a hydrodynamic system's sensitivity to initial conditions, the concept of deterministic chaos became a dominant concept of weather and climate theory. Our findings suggest that if we consider orbital time scales and, instead of time series, focus on their more generalized attributes such as the period of the system response to the astronomical forcing, we may observe that the behavior of these attributes may be, to some extent, less sensitive to the physical nature of the terrestrial governing processes.

But is incomplete similarity of the global, orbital-scale, climate system real? So far, this property has been found only in our VCV18 low-order dynamical model, and although this model has been explicitly derived from the conservation laws, the incomplete similarity of the ice-climate system will remain hypothetical until it is supported by empirical data. We speculate, though, that existing historical records may provide some support to this concept. To evaluate the feasibility of a diagnostic approach, let us entertain a simple scaling exercise. Suppose that an empirical time series, such as $\delta^{18}O$ record, is created by a parent system (other than the VCV18) which is controlled by $n$ physical parameters ($k$ of them having independent dimensions). If we choose the period of the astronomical forcing $T$ to be among parameters with independent dimensions, then in accordance with the π-theorem we have:

$$P = T\Psi(\pi_1, \pi_2, \dots, \pi_{n-k}) \quad (9)$$

The wavelet spectrum of the late Pleistocene $\delta^{18}O$ variability in response to the precession (~20-kyr period) and obliquity (~40-kyr period) forcing shows the dominance of 40-kyr and 80-kyr periods (Fig. 1c). If we are willing to accept it as a hint of $\Psi = 2$ for $T$ = 20 kyr and for $T$ = 40 kyr, then, since some of the similarity parameters $\pi_1, \pi_2, \dots, \pi_{n-k}$ depend on $T$, the period-$T$ independence of $\Psi$ may only happen when $\pi_1, \pi_2, \dots, \pi_{n-k}$ make conglomerate $T$-independent groups. In other words, *period independence of the $\Psi$ function may be a signature of climate system incomplete similarity*. Indeed, the diagnostics of the $\Psi$ function may require much more sophisticated instruments than our *ad hoc* reasoning, and the records will likely not explicitly reveal what the conglomerate similarity groups look like; nevertheless, their mere existence would corroborate the idea of this paper.

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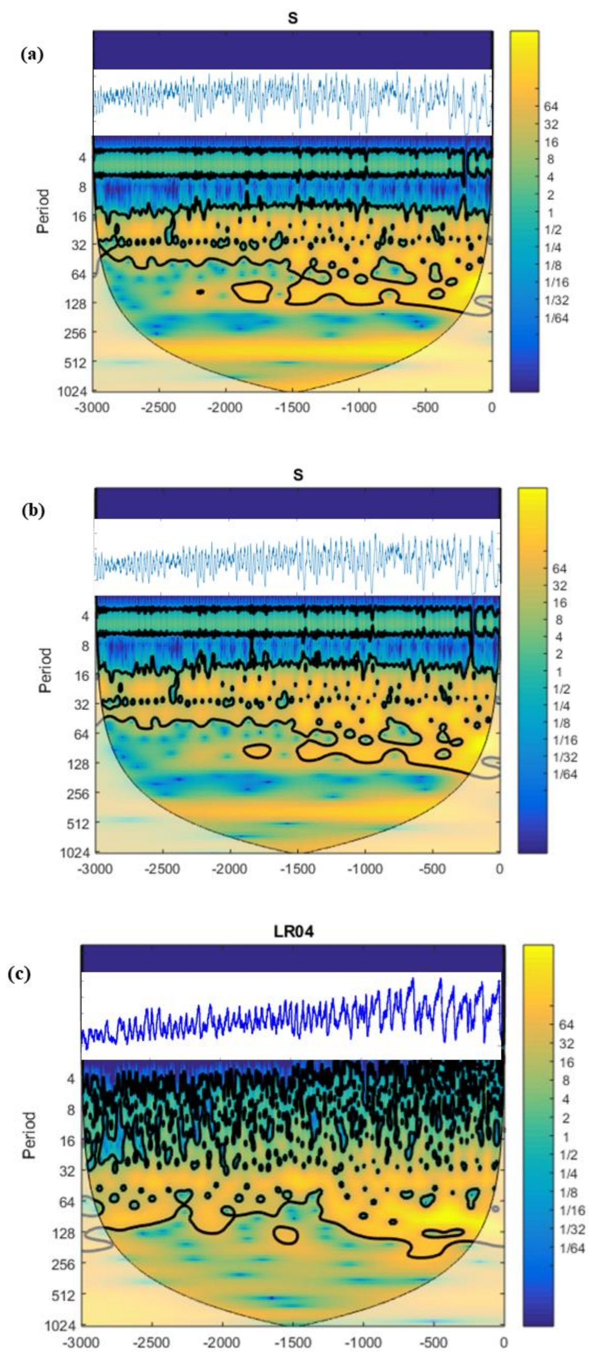


**Fig. 1** Ice-climate system response to a cooling trend presented as an evolution of wavelet spectra over 3 Myr for calculated ice-sheet glaciation area *S* ($10^6$ km$^2$) – panels (a) and (b), and for the Lisiecki and Raymo (2005) benthic $\delta^{18}$O record, panel (c). The *V*-number evolves from *V* = 0.5 to *V* = 0.75 due to weakening of the negative feedback (a) and due to intensified positive feedback (b). The vertical axis is the period (kyr), the horizontal axis is time (kyr before present). The color scale shows the continuous Morlet wavelet amplitude, the thick line indicates the peaks with 95 % confidence, and the shaded area indicates the cone of influence for wavelet transform. Inserts are corresponding time series.