Supplementary material to Climate change in a conceptual atmosphere–plankton model
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Supplementary Material I: Analytic results without mixing

We derive analytic results for the long-term concentration and temperature contrast behavior when mixing is negligible ($\gamma = 0$). Numbers in (brackets) refer to the equations of the main part paper, letter S and numbers refer to equations on this Supplamantary Material.

Without seasonality

Let us consider equation (2) and (9) with $\gamma = 0$ and $A = 0$ leading to

$$\dot{c} = rc\left(1 - \frac{c}{1-\alpha(-D_0 t + \beta(c(t)-1))}\right). \tag{S1}$$

This equation is found to possess an asymptotically linear behavior

$$c^*(t) = St + \delta. \tag{S2}$$

With this form, from (4) and (8), we find

$$F(t) - F_r = -D_0 t + \beta S t + \beta(\delta - 1) \equiv -Dt + \beta(\delta - 1), \tag{S3}$$

where

$$D = D_0 - \beta S \tag{S4}$$

is the slope of the time evolution of the temperature contrast. This shows that the temperature contrast, and in particular, the strength of the climate change, becomes *influenced* by the phytoplankton concentration.

In order to specify the unknown constants S and δ, let us rewrite (S1) as

$$S\left(\alpha D t + 1 - \alpha\beta(\delta - 1)\right) = r(St + \delta)\left[(\alpha D - S)t + 1 - \alpha\beta(\delta - 1) - \delta\right]. \tag{S5}$$

For long times, i.e. $t \gg 1/r$, the quadratic term dominates on the right hand side which cannot be compensated by anything on the left hand side. The coefficient of the quadratic term should vanish, i.e. $\alpha D = S$ from which, since $D = D_0 - \beta S$,

$$S = \frac{D_0 \alpha}{1 + \beta\alpha}, \tag{S6}$$

$$D = \frac{D_0}{1 + \beta\alpha}, \tag{S7}$$

as stated in (11,12).

Assuming that the linear form is valid not only for very large times, but for intermediate ones, too, the linear terms on both sides should compensate each other. From (S5), this yields

$$S\alpha D = rS\left[1-\alpha\beta(\delta-1)-\delta\right]. \tag{S8}$$

After dividing by S and using that $\alpha D = S$, we find

$$S = r(1-\delta(1+\alpha\beta)+\alpha\beta) \tag{S9}$$

from which

$$\delta = \frac{1+\alpha\beta - S/r}{1+\alpha\beta} = \frac{1+\alpha\beta-\alpha D/r}{1+\alpha\beta}. \tag{S10}$$

This result provides the constant δ in the long-term dynamics (S2) of phytoplankton.

With seasonality

Let us add now periodic forcing in the form of (4) with (8), yielding

$$F(t) = F_r - D_0 t + \beta(c(t)-1) + A\sin(\omega t). \tag{S11}$$

We expect the phytoplankton concentration to be driven to oscillate with the same frequency ω but with some phase shift ϕ. For long times, $t \gg 1/r$, we assume that the snapshot attractor is of the form

$$c^*(t) = St + \delta + B\sin(\omega t - \phi) \tag{S12}$$

with an amplitude B. The corresponding long term behavior of the carrying capacity is

$$K(t) = 1 - \alpha(-Dt + \beta(\delta-1) + A\sin(\omega t) + \beta B\sin(\omega t - \phi)). \tag{S13}$$

In order to fix the constants, we substitute these into (2) to find

$$(S + B\omega\cos(\omega t-\phi))\left[\alpha D t + 1 - \alpha(\beta(\delta-1) + A\sin(\omega t) + \beta B\sin(\omega t-\phi))\right] =$$

$$r(St+\delta+B\sin(\omega t-\phi))\left[(\alpha D-\gamma)t + 1 - \alpha(\beta(\delta-1)+A\sin(\omega t)) - \delta - B(1+\beta\alpha)\sin(\omega t-\phi)\right]. \tag{S14}$$

The vanishing of the quadratic term in t provides the same equation as without periodic forcing, therefore (S6) and (S7) turn out to remain valid. Similarly, from the linear terms (S10) is recovered.

From the product of trigonometric and linear terms, the following equation follows:

$$B\omega D\alpha\cos(\omega t-\phi) = BS\omega\cos(\omega t-\phi)$$

$$= rS(-\alpha A\sin(\omega t) - B(1+\alpha\beta)\sin(\omega t-\phi)).$$

After an application of trigonometric identities, from the vanishing of the coefficient of $t\cos(\omega t)$ we find

$$\tan\phi = \frac{\omega}{r(1+\beta\alpha)}. \tag{S15}$$

From the vanishing of the coefficient of $t\sin(\omega t)$

$$B[\omega \sin\phi + r(1+\beta\alpha)\cos\phi] = -r\alpha A,$$

from which the concentration amplitude is found to be

$$B = -\frac{\alpha A}{\sqrt{(1+\beta\alpha)^2 + \omega^2/r^2}}. \tag{S16}$$

We have thus been able to determine analytically the snapshot attractor $c^*(t)$ in a strongly nonlinear model with a linear drift and periodic forcing. The snapshot attractor in the concentration remains point-like (no internal variability in c), but changes in an oscillatory fashion about a linear growth with an amplitude B determined by all the system parameters.

Supplementary Material II: Figures

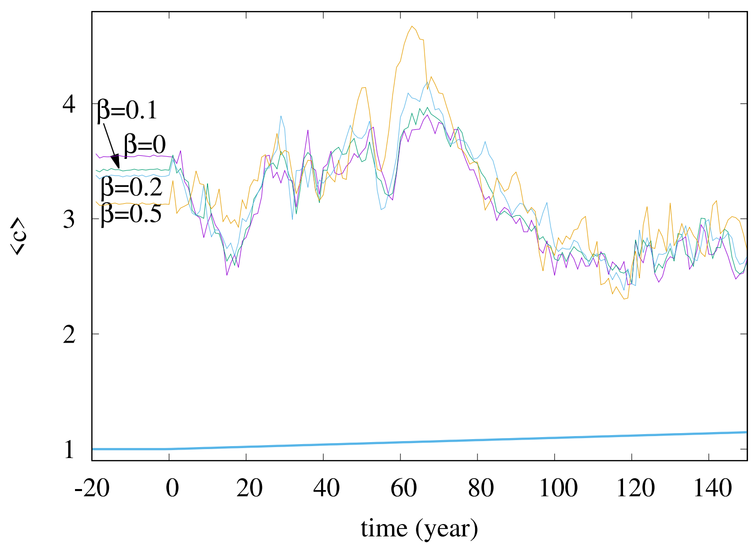

Figure S1. Average phytoplankton concentration $\langle c \rangle$ as a function of time for $\alpha = 0.05$, $\gamma = 0.1$ for various values of β. The straight blue line shows the expected phytoplankton concentration in lack of mixing ($\gamma = 0$) as predicted by Eqs. (S2), (S6), (S10).

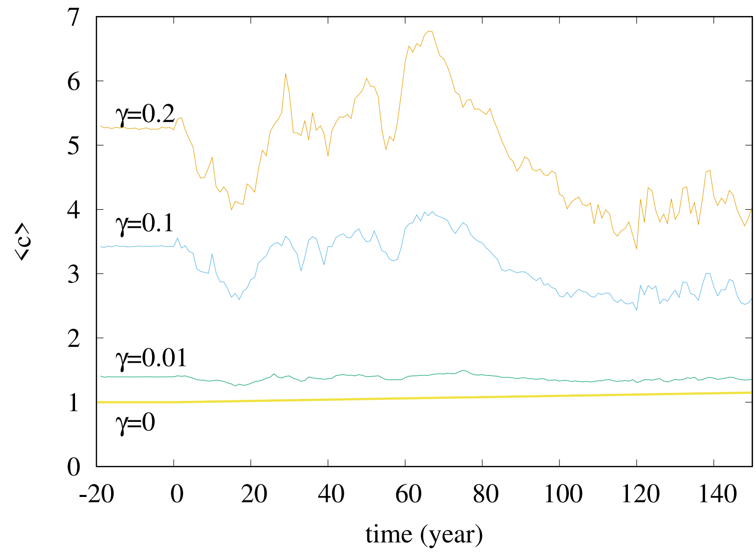

Figure S2. Average phytoplankton concentration $\langle c \rangle$ as a function of time for $\alpha = 0.05$, $\beta = 0.1$ for various values of γ ($\gamma = 0, 0.01, 0.1$ and 0.2). The straight yellow line shows the expected phytoplankton concentration in the lack mixing ($\gamma = 0$) as predicted by Eqs. (S2), (S6), (S10).

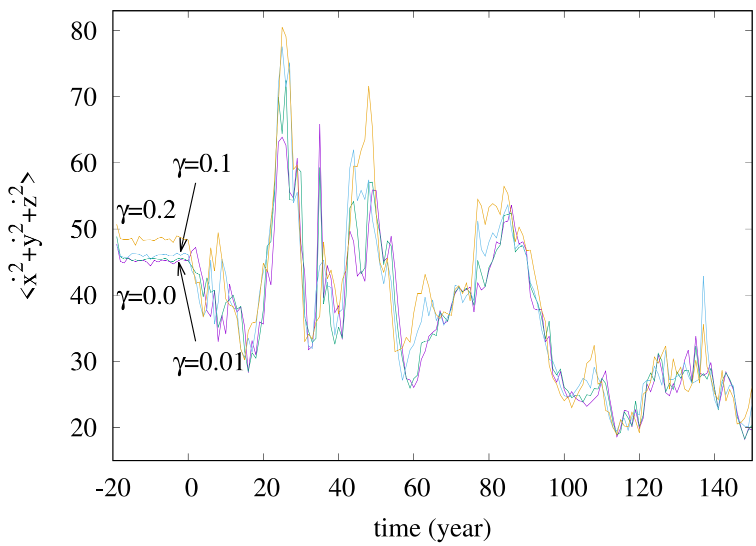

Figure S3. Time-dependence of the ensemble average (over 50000 realization) of the total atmospheric kinetic energy $\dot{x}^2 + \dot{y}^2 + \dot{z}^2$ for $\beta = 0.1$, $\alpha = 0.05$ and $\gamma = 0, 0.01, 0.1, 0.2$. The curves for different values of γ are quite similar.