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Fractional governing equations of transient groundwater flow in unconfined aquifers with multi-fractional dimensions in fractional time

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Abstract: In this study, a dimensionally-consistent governing equation of transient unconfined groundwater flow in fractional time and multi-fractional space is developed. First, a fractional continuity equation for transient unconfined groundwater flow is developed in fractional time and space. For the equation of groundwater motion within a multi-fractional multi-dimensional unconfined aquifer, a previously-developed dimensionally consistent equation for water flux in unsaturated/saturated porous media is combined with the Dupuit approximation to obtain an equation for groundwater motion in multi-fractional space in unconfined aquifers. Combining the fractional continuity and groundwater motion equations, the fractional governing equation of transient unconfined aquifer flow is then obtained. Finally, a numerical application to an unconfined aquifer groundwater flow problem is presented to show the skills of the proposed fractional governing equation.

1. Introduction

One way to obtain non-Fickian behavior in solute transport is by treating the underlying flow field to have long-range dependence in time (Kim et al., 2015; Kavvas et al., 2015). As shown by

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Ercan and Kavvas (2014, and 2017), such dependence in time can be modeled by a time-fractional governing equation of the specified flow field. Flow velocity correlation and distribution in fractured media, which can be modeled by Continuous Time Random Walk models (Metzler and Klafter, 2000), may also result in non-Fickian transport (Kang et al., 2015). Long-range dependence in time reported in groundwater level fluctuations (e.g., Li and Zhang, 2007; Yu et al., 2016; Tu et al., 2017; and the references therein) and anisotropy in aquifer medium necessitates time- and space-fractional operators in the governing equations of groundwater flow (Kavvas et al., 2017a).

Reporting that conventional geometries cannot characterize groundwater flow in many fractured rock aquifers (Black et al., 1986), and the observed drawdown tends to be underestimated in early times and overestimated at later times by the conventional radial groundwater flow model (Van Tonder et al., 2001), Cloot and Botha (2006) developed a fractional governing equation for radial groundwater flow in integer time and fractional space in a uniform homogeneous aquifer. They used the Riemann-Liouville fractional derivative form in the model formulation. Atangana and Bildik (2013), Atangana (2014), and Atangana and Vermeulen (2014) then reformulated the fractional radial groundwater flow model of Cloot and Botha (2006) by the Caputo differentiation framework, and reported better performance. Compared to the Riemann-Liouville derivative approach, the Caputo framework has a fundamental advantage of being able to accommodate physically-interpretable real-life initial and boundary conditions (Podlubny, 1998). Atangana and Baleanu (2014) presented a new radial groundwater flow model in fractional time based on a new fractional derivative definition, "conformable derivative" (Khalil et al., 2014). Most recently, Su (2017) proposed a time-space fractional Boussinesq equation and he claimed this fractional equation is a general groundwater flow equation and can be applied to groundwater flow in both confined and unconfined aquifers. However, all of the aforementioned studies only presented the formulated fractional governing groundwater flow equations and no detailed derivations of these governing equations from the fundamental conservation principles were provided.

Wheatcraft and Meerschaert (2008) derived the groundwater flow continuity equation in the fractional form by using the fractional Taylor series approximation. They further removed the linearity / piecewise linearity restriction for the flux and the infinitesimal control volume restriction. When developing the fractional continuity equation, the groundwater flow process was considered in fractional space but in integer time by Wheatcraft and Meerschaert (2008). They

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further assumed the same fractional power in every direction of the fractional porous media space. Furthermore, only the mass conservation was considered in their derivation, but not the fractional water flux equation. Mehdinejadiani et al. (2013) expanded the approach of Wheatcraft and Meerschaert (2008) to the derivation of a governing equation of groundwater flow in an unconfined aquifer in fractional space but in integer time. In their derivation, they used the conventional Darcy formulation for the water flux with integer spatial derivative while utilizing fractional spatial derivatives in their continuity equation.

Olsen et al. (2016) pointed out that the derivations in Wheatcraft and Meerschaert (2008) and Mehdinejadiani et al. (2013) utilized the fractional Taylor series, as formulated by Odibat and Shawagfeh (2007), which utilized local Caputo derivatives. In order to expand the local Caputo derivatives in the above-mentioned studies, Olsen et al. (2016) utilized the fractional mean value theorem from Diethelm (2012) to develop a continuity equation of groundwater flow with left and right fractional nonlocal Caputo derivatives in fractional space but in integer time. Olsen et al. (2016) did not address the water flux formulation in fractional space, and, hence, did not develop a complete governing equation of groundwater flow. They also did not address the multifractional spatial derivatives in order to address anisotropy within an aquifer. Around that time, Kavvas et al. (2017a) utilized the mean value formulation from Usero (2007), Odibat and Shawagfeh (2007) and Li et al. (2009) to derive a complete governing equation of transient groundwater flow in a confined, anisotropic aquifer with fractional time and multi-fractional space derivatives which addressed not only the continuity but also the water flux (motion) in fractional time-space and the effect of a sink/source term. By employing the above-mentioned fractional mean value formulations, Kavvas et al. (2017a) developed the governing equation of confined groundwater flow in fractional time-space in non-local form.

Unconfined groundwater flow is the fundamental component of the watershed runoff baseflow since it is the fundamental contributor to the network streamflow within a watershed during dry periods. As such, the behavior of unconfined groundwater flow is key to the physically-based understanding of the long memory in watershed runoff. Meanwhile, as will be seen in the following derivation of its governing equation, unconfined aquifer groundwater flow is uniquely different from the confined aquifer groundwater flow. The fundamental differences between the two aquifer flows is that while the flow in a confined aquifer is linear and compressible, the flow in an unconfined aquifer is nonlinear and incompressible due to the unconfined aquifer being

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phreatic, its top surface boundary being open to the atmosphere. Accordingly, hydrologists have developed unique governing equations of unconfined aquifer groundwater flow (Bear, 1979; Freeze and Cherry, 1979). Starting with the next section, first the continuity equation of transient unconfined groundwater flow within an anisotropic heterogeneous aquifer under a time-space varying sink/source will be developed in fractional time and fractional space. Then, this fractional continuity equation will be combined with a fractional groundwater motion equation to obtain a transient groundwater flow equation in fractional time-multifractional space within an anisotropic, heterogeneous unconfined aquifer.

Analogous to the traditional governing groundwater flow equations, as outlined by Freeze and Cherry (1979) and Bear (1979), the fractional unconfined groundwater flow equations must have specific features (Kavvas et al., 2017a):

i. In order for the governing equation to be prognostic, the form of the equation must be known completely from the outset.

ii. The fractional governing equations must be dimensionally consistent and be purely differential equations, containing only differential operators without difference operators.

iii. As the fractional derivative powers go to integer values, the fractional unconfined groundwater flow equations must converge to the corresponding conventional integer-order governing equations.

Within this framework, the governing equations of unconfined groundwater flow in fractional time and fractional space will be developed in the following.

2. Derivation of the Continuity Equation for Transient Unconfined Groundwater Flow in a Heterogeneous Anisotropic Multi-Fractional Medium in Fractional Time

To β-order the Caputo fractional derivative $D_a^{k\beta} f(x)$ of a function *f(x)* may be defined as (Odibat and Shawagfeh, 2007; Podlubny, 1998; Usero, 2007, Li et al., 2009),

$$D_a^{\beta} f(x) = \frac{1}{\Gamma(1-\beta)} \int_a^x \frac{f^{`}(\xi)}{(x-\xi)^{\beta}} d\xi \qquad 0 < \beta < 1, \quad x \geq a \quad . \tag{1}$$

It was shown in Kavvas et al. (2017b) that one can obtain a β_{x_i}-order approximation (i=1,2) to a function $f(x_i)$ around x_i - Δx_i as

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$$f(x_i) = f(x_i - \Delta x_i) + \frac{(\Delta x_i)^{\beta_{x_i}}}{\Gamma\left(\beta_{x_i}+1\right)} D_{x_i-\Delta x_i}^{\beta_{x_i}} f(x_i) \quad ; \text{i=1,2.} \tag{2}$$

In Equation (2), an analytical relationship between Δx_i and $(\Delta x_i)^{\beta_{x_i}}$ (i=1,2) that will be universally applicable throughout the modelling domain is possible when the lower limit in the above Caputo derivative in equation (2) is taken as zero (that is, $\Delta x_i = x_i$) for f(x_i) =x_i (Kavvas et al. 2017b).

Under the Dupuit approximation of horizontal flow streamlines (very small water table gradient) (Bear, 1979), the net mass flux through the control volume of an unconfined aquifer with a flat bottom confining layer, as depicted in Figure 1, that also has a sink/source mass flux $\rho q_v \Delta x_1 \Delta x_2$, can be formulated as

$$\left[\rho Q_{x_1}(x_1, x_2; t) - \rho Q_{x_1}(x_1 - \Delta x_1, x_2; t)\right]\Delta x_2 + \left[\rho Q_{x_2}(x_1, x_2; t) - \rho Q_{x_2}(x_1, x_2 - \Delta x_2; t)\right]\Delta x_1 - \rho q_v \Delta x_1 \Delta x_2 \tag{3}$$

where Q_{x_i} is the discharge across a vertical plane of unit width in i-th direction, i = 1,2, ρ is the fluid density, and q_v is the source/sink (recharge/leakage) per unit horizontal area. Then by combining equation (2) with equation (3) with $\Delta x_i = x_i$ (i=1,2), and expressing the resulting Caputo derivative $D_0^{\beta_{x_i}} f(x_i)$ by $\frac{\partial^{\beta_{x_i}} f(x_i)}{(\partial x_i)^{\beta_{x_i}}}$, (i=1,2) for convenience, yields the net mass flux through the control volume in Figure 1 to the orders of $(\Delta x_1)^{\beta_{x_1}}$ and $(\Delta x_2)^{\beta_{x_2}}$, as

$$\frac{1}{\Gamma(\beta_{x_1}+1)}\left(\frac{\partial}{\partial x_1}\right)^{\beta_{x_1}}\left(\rho Q_{x_1}(x_1, x_2; t)\right)(\Delta x_1)^{\beta_{x_1}}\Delta x_2 + \frac{1}{\Gamma(\beta_{x_2}+1)}\left(\frac{\partial}{\partial x_2}\right)^{\beta_{x_2}}\left(\rho Q_{x_2}(x_1, x_2; t)\right)\Delta x_1 (\Delta x_2)^{\beta_{x_2}} - \rho q_v \Delta x_1 \Delta x_2 \tag{4}$$

where different powers for fractional space derivatives are utilized in different directions due to the anisotropy in the flow medium.

Kavvas et al. (2017b) have shown that to β_{x_i}-order fractional increments in space in the i-th direction, i=1,2,

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$$(\Delta x_i)^{\beta_{x_i}} = \frac{\Gamma\left(\beta_{x_i}+1\right)\Gamma(2-\beta_{x_i})}{{x_i}^{1-\beta_{x_i}}}\,\Delta x_i \qquad , \text{ i=1,2.} \tag{5}$$

Combining equations (5) and (4) yields for the net mass outflow through the control volume in Figure 1 as (to the order of $(\Delta x_i)^{\beta_{x_i}}$, i=1,2),

$$\frac{\Gamma(2-\beta_{x_1})}{{x_1}^{1-\beta_{x_1}}}\left(\frac{\partial}{\partial x_1}\right)^{\beta_{x_1}}\left(\rho Q_{x_1}(\bar{x};t)\right)\Delta x_1\Delta x_2 + \frac{\Gamma(2-\beta_{x_2})}{{x_2}^{1-\beta_{x_2}}}\left(\frac{\partial}{\partial x_2}\right)^{\beta_{x_2}}\left(\rho Q_{x_2}(\bar{x};t)\right)\Delta x_1\Delta x_2 - \rho q_v \Delta x_1\Delta x_2, \quad \bar{x} = (x_1\,, x_2). \tag{6}$$

Denoting the water volume within the control volume in Figure 1 by V_w and using the concept of specific yield (effective porosity) S_y of a phreatic aquifer (Bear and Verruijt, 1987)

$$S_y = \frac{\Delta V_w}{\Delta h}\frac{1}{\Delta x_1 \Delta x_2} \qquad , \tag{7}$$

where ΔV_w is the change in water volume in the control volume per change Δh in the hydraulic head (the elevation of the phreatic surface (water table) above the flat bottom of the aquifer), the time rate of change of mass within the control volume in Figure 1 may be written as (Bear and Verruijt, 1987)

$$\frac{S_y\left(\rho h(\bar{x};t) - \rho h(\bar{x};t-\Delta t)\right)}{\Delta t}\,\Delta x_1 \Delta x_2 \tag{8}$$

which can then be expressed in terms of the approximation (2) with respect to the time dimension as,

$$\frac{S_y}{\Delta t}\left[\frac{\Delta t^{\alpha}}{\Gamma(\alpha+1)}\left(\frac{\partial}{\partial t}\right)^{\alpha}(\rho h)\right]\Delta x_1 \Delta x_2 \qquad . \tag{9}$$

To α-order fractional increments in time (Kavvas et al. 2017b)

$$(\Delta t)^{\alpha} = \frac{\Gamma(\alpha+1)\Gamma(2-\alpha)}{t^{1-\alpha}}\,\Delta t \qquad . \tag{10}$$

Substituting equation (10) into equation (9), one can obtain the time rate of change of mass in the control volume, as shown in Figure 1;

$$S_y \frac{\Gamma(2-\alpha)}{t^{1-\alpha}} \left(\frac{\partial}{\partial t}\right)^{\alpha} (\rho h)\, \Delta x_1 \Delta x_2 \,. \tag{11}$$

As the time rate of change of mass within the control volume, as shown in Figure 1, must be inversely proportional to the net mass flux passing through the control volume, one may combine Equations (6) and (11) to obtain

$$\left[\frac{\Gamma(2-\beta_{x_1})}{{x_1}^{1-\beta_{x_1}}} \left(\frac{\partial}{\partial x_1}\right)^{\beta_{x_1}} \left(\rho Q_{x_1}(\bar{x};t)\right) + \frac{\Gamma(2-\beta_{x_2})}{{x_2}^{1-\beta_{x_2}}} \left(\frac{\partial}{\partial x_2}\right)^{\beta_{x_2}} \left(\rho Q_{x_2}(\bar{x};t)\right) - \rho q_v\right] \Delta x_1 \Delta x_2 =$$
$$- S_y \frac{\Gamma(2-\alpha)}{t^{1-\alpha}} \left(\frac{\partial}{\partial t}\right)^{\alpha} (\rho h)\, \Delta x_1 \Delta x_2 \tag{12}$$

$$\frac{\Gamma(2-\beta_{x_1})}{{x_1}^{1-\beta_{x_1}}} \left(\frac{\partial}{\partial x_1}\right)^{\beta_{x_1}} \left(\rho Q_{x_1}(\bar{x};t)\right) + \frac{\Gamma(2-\beta_{x_2})}{{x_2}^{1-\beta_{x_2}}} \left(\frac{\partial}{\partial x_2}\right)^{\beta_{x_2}} \left(\rho Q_{x_2}(\bar{x};t)\right) - \rho q_v = - S_y \frac{\Gamma(2-\alpha)}{t^{1-\alpha}} \left(\frac{\partial}{\partial t}\right)^{\alpha} (\rho h) \tag{13}$$

for $0 < \alpha, \beta_{x_1}, \beta_{x_2} < 1$, $\bar{x} = (x_1\,, x_2,)$.

Within the framework of fluid incompressibility in the unconfined aquifer, equation (13) reduces further to

$$\frac{\Gamma(2-\beta_{x_1})}{{x_1}^{1-\beta_{x_1}}} \left(\frac{\partial}{\partial x_1}\right)^{\beta_{x_1}} \left(Q_{x_1}(\bar{x};t)\right) + \frac{\Gamma(2-\beta_{x_2})}{{x_2}^{1-\beta_{x_2}}} \left(\frac{\partial}{\partial x_2}\right)^{\beta_{x_2}} \left(Q_{x_2}(\bar{x};t)\right) - q_v = - S_y \frac{\Gamma(2-\alpha)}{t^{1-\alpha}} \frac{\partial^{\alpha} h}{(\partial t)^{\alpha}}$$

$$\frac{\Gamma(2-\beta_{x_1})}{\Gamma(2-\alpha)} \frac{t^{1-\alpha}}{{x_1}^{1-\beta_{x_1}}} \left(\frac{\partial}{\partial x_1}\right)^{\beta_{x_1}} \left(Q_{x_1}(\bar{x};t)\right) + \frac{\Gamma(2-\beta_{x_2})}{\Gamma(2-\alpha)} \frac{t^{1-\alpha}}{{x_2}^{1-\beta_{x_2}}} \left(\frac{\partial}{\partial x_2}\right)^{\beta_{x_2}} \left(Q_{x_2}(\bar{x};t)\right) - \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} q_v$$
$$= - S_y \frac{\partial^{\alpha} h}{(\partial t)^{\alpha}} \tag{14}$$

for $0 < \alpha, \beta_{x_1}, \beta_{x_2} < 1$, $\bar{x} = (x_1\,, x_2,)$ as the time-space fractional continuity equation of transient groundwater flow in an anisotropic unconfined aquifer with multi-fractional dimensions and in fractional time.

Performing a dimensional analysis of Equation (14) yields

$$\frac{L}{T^{\alpha}} = \frac{T^{1-\alpha}}{L^{1-\beta_{x_1}}} \frac{1}{L^{\beta_{x_1}}} \frac{L^2}{T} = \frac{T^{1-\alpha}}{L^{1-\beta_{x_2}}} \frac{1}{L^{\beta_{x_2}}} \frac{L^2}{T} = \frac{T^{1-\alpha}}{1} \frac{L}{T} = \frac{1}{L^{1-\beta_z}} \frac{1}{L^{\beta_z}} \frac{L}{T} = \frac{L}{T^{\alpha}} \tag{15}$$

where L denotes length and T denotes time. Hence, the left-hand and right-hand sides of the continuity equation (14) for transient groundwater flow in an unconfined aquifer in multi-fractional space and fractional time are consistent as shown in equation (15).

For n-1< $\alpha, \beta_{x_i} < n$ where n is any positive integer, as α and $\beta_{x_i} \rightarrow$ n, the Caputo fractional derivative of a function f(y) to order α or β_{x_i} (i = 1, 2) yields the standard n-th derivative of the function f(y) (Podlubny, 1998). When α and $\beta_{x_i} \rightarrow 1$ (i = 1, 2), the continuity equation (14) becomes the conventional continuity equation for transient groundwater flow in an unconfined aquifer:

$$-S_y \frac{\partial h}{\partial t} = \frac{\partial}{\partial x_1}\left(Q_{x_1}(\bar{x};t)\right) + \frac{\partial}{\partial x_2}\left(Q_{x_2}(\bar{x};t)\right) - q_v \ . \qquad (16)$$

3. Motion Equation (Specific Discharge) in Fractional Multi-Dimensional Unconfined Aquifers

Recently, Kavvas et al., (2017a, 2017b) derived a governing equation for water flux (specific discharge), q_{x_i}, (i = 1, 2, 3) in a saturated or unsaturated porous medium with fractional dimensions in the form,

$$q_i(\bar{x},t) = -K_{s,x_i}(\bar{x}) \frac{\Gamma(2-\beta_{x_i})}{x_i^{\,1-\beta_{x_i}}} \frac{\partial^{\beta_{x_i}} h}{(\partial x_i)^{\beta_{x_i}}} \ , \ \mathrm{i} = 1,2,3; \quad \bar{x} = (x_1\,, x_2, x_3\,). \qquad (17)$$

where $K_{s,x_i}(\bar{x})$ is the saturated hydraulic conductivity in the i-th spatial direction (i=1,2,3). Meanwhile, under the Dupuit approximation of essentially horizontal unconfined aquifer flow (water table slope very small) (Bear, 1979), referring to Figure 1, the discharge per unit width in the i-th direction (i = 1,2) can be expressed as

$$Q_{x_i}(\bar{x},t) = hq_i(\bar{x},t), \ \mathrm{i} = 1,2 \quad ; \quad \bar{x} = (x_1\,, x_2,). \qquad (18)$$

Then combining equations (18) and (17) results in

$$Q_{x_i}(\bar{x},t) = -K_{s,x_i}(\bar{x}) \frac{\Gamma(2-\beta_{x_i})}{x_i^{\,1-\beta_{x_i}}} h \frac{\partial^{\beta_{x_i}} h}{(\partial x_i)^{\beta_{x_i}}} \quad , \ \mathrm{i} = 1,2; \ \bar{x} = (x_1\,, x_2,) \qquad (19)$$

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as the governing equation of groundwater motion within an unconfined aquifer with a flat bottom confining layer. In equation (19) h is the unconfined aquifer thickness or the phreatic surface elevation above the bottom confining layer.

A dimensional analysis on equation (19) yields L^2/T for the units of both the left-hand-side (LHS) and the RHS of the equation, establishing its dimensional consistency.

Applying the above-mentioned result of Podlubny (1998) on the convergence of a fractional derivative to a corresponding integer derivative for $\beta_{x_i} \rightarrow 1$ (i = 1, 2) reduces the fractional motion equation (19) for unconfined groundwater flow to the conventional equation (Bear, 1979):

$$Q_{x_i}(\bar{x},t) = -K_{s,x_i}(\bar{x})\, h \frac{\partial h(\bar{x},t)}{\partial x_i}, \text{ i}= 1,2 \tag{20}$$

for the case of integer spatial dimensions. As such, the fractional motion equation (19) for unconfined groundwater flow in fractional spatial dimensions is consistent with the conventional motion equation for the integer spatial dimensions.

4. The Complete Equation for Transient Unconfined Groundwater Flow in Multi-Fractional Space and Fractional Time

Combining the fractional motion equation (19) of groundwater flow in an unconfined aquifer with the fractional continuity equation (14) of unconfined groundwater flow results in the equation,

$$S_y \frac{\partial^\alpha h}{(\partial t)^\alpha} = \frac{\Gamma(2-\beta_{x_1})}{{x_1}^{1-\beta_{x_1}}} \left(\frac{\partial}{\partial x_1}\right)^{\beta_{x_1}} \left(K_{s,x_1}(\bar{x}) \frac{t^{1-\alpha}}{{x_1}^{1-\beta_{x_1}}} \frac{\Gamma(2-\beta_{x_1})}{\Gamma(2-\alpha)}\, h \frac{\partial^{\beta_{x_1}} h}{(\partial x_1)^{\beta_{x_1}}}\right) + \frac{\Gamma(2-\beta_{x_2})}{{x_2}^{1-\beta_{x_2}}} \left(\frac{\partial}{\partial x_2}\right)^{\beta_{x_2}} \left(K_{s,x_2}(\bar{x}) \frac{t^{1-\alpha}}{{x_2}^{1-\beta_{x_2}}} \frac{\Gamma(2-\beta_{x_2})}{\Gamma(2-\alpha)}\, h \frac{\partial^{\beta_{x_2}} h}{(\partial x_2)^{\beta_{x_2}}}\right) + \frac{t^{1-\alpha}}{\Gamma(2-\alpha)}\, q_v \tag{21}$$

for $0 < \alpha, \beta_{x_1}, \beta_{x_2} < 1$, $\bar{x} = (x_1, x_2,)$ as the time-space fractional governing equation of transient unconfined groundwater flow in an anisotropic medium.

Performing a dimensional analysis of Equation (21) yields

$$\frac{L}{T^\alpha} = \frac{1}{L^{1-\beta_{x_1}}} \frac{1}{L^{\beta_{x_1}}} \frac{L}{T} \frac{T^{1-\alpha}}{L^{1-\beta_{x_1}}}\, L \frac{L}{L^{\beta_{x_1}}} = \frac{1}{L^{1-\beta_{x_2}}} \frac{1}{L^{\beta_{x_2}}} \frac{L}{T} \frac{T^{1-\alpha}}{L^{1-\beta_{x_2}}} \frac{L^2}{L^{\beta_{x_2}}} = \frac{T^{1-\alpha}}{1} \frac{L}{T} = \frac{L}{T^\alpha} \tag{22}$$

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where L denotes length and T denotes time. Hence, the left-hand and right-hand sides of the governing equation (21) for transient groundwater flow in an unconfined aquifer in multi-fractional space and fractional time are consistent.

Specializing the above-discussed result of Podlubny (1998) to n = 1, for α and $\beta_{x_i} \rightarrow 1$ (i = 1, 2) reduces the governing fractional equation (21) to the conventional governing equation for transient groundwater flow in an unconfined aquifer (Bear, 1979):

$$S_y \frac{\partial h}{\partial t} = \frac{\partial}{\partial x_1}\left(K_{s,x_1}(\bar{x}) h \frac{\partial h(\bar{x},t)}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(K_{s,x_2}(\bar{x}) h \frac{\partial h(\bar{x},t)}{\partial x_2} \right) + q_v \tag{23}$$

5. Numerical application

To demonstrate the skills of the proposed fractional governing equation of unconfined aquifer groundwater flow, a numerical application is performed using the proposed fractional governing equation to the physical setting of an example from Wang and Anderson (1995), as depicted in Figure 2. The numerical problem of seepage through a dam under a sudden change in the water surface elevation at the downstream section of the dam is modified based on seepage through a dam, Page 53 and Problem 4.4 (a), Page 89 in Wang and Anderson (1995), as shown in Figure 2. The water seepage through the dam's body may be interpreted as one-dimensional groundwater flow through an unconfined aquifer. The unconfined flow system locates the top boundary of the saturated zone in an earthen dam and the bottom of the dam rests on impermeable rock. For this example, the unconfined aquifer length L is 100 m. The initial water level in the upstream and downstream sections of the dam and through the dam's body is 16 m. Then immediately after the initial time, the water level at the downstream section of the dam is suddenly dropped to 11 m and remains as 11 m afterwards. The unconfined aquifer parameters, storage coefficient and hydraulic conductivity, are S = 0.2, K=0.002 m/min respectively. The analytical solution for this problem at the steady-state is:

$$h = \sqrt{\frac{h_2^2 - h_1^2}{L} x + h_1^2} \tag{24}$$

where h is the depth of the unconfined groundwater surface from the bottom layer; L is the aquifer length; x is the distance from the upstream location of the dam body, and h_1 and h_2 are as shown in Figure 2.

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In Figure 3(a) the normalized groundwater head h/h_1 at location $x=L/2$ through time under different fractional power values is shown. Meanwhile, Figure 3(b) shows the normalized groundwater head h/h_1 at the time instance t=15000 min as function of location throughout the dam's body, and the analytical solution of the standard governing equation of unconfined groundwater flow when $\beta_x = \alpha = 1$ at the steady state. The considered fractional derivative powers in space and time are $\beta_x = \alpha = 0.7, 0.8, 0.9, 1.0$. As can be seen from Figure 3(a), the hydraulic head recession in time slows down with the decrease of $\beta_x = \alpha$ from 1. The hydraulic heads in Figure 3(a) have heavier tails as orders of time and space fractional derivatives decrease from 1 towards 0.7. Meanwhile, Figure 3(b) shows that the numerical solution of the governing fractional equation at $\beta_x = \alpha = 1.0$ and at a very long time after the initial condition, matches perfectly the steady state analytical solution (24) of the standard equation (23) with the specified initial/boundary conditions.

6. Discussion

From the standard governing equation (23) of unconfined groundwater flow in integer time-space the saturated hydraulic conductivity may be interpreted as a diffusion coefficient for the nonlinear diffusion of groundwater in an unconfined aquifer. The basic difference between confined and unconfined groundwater flow is that the former can be interpreted as a linear diffusion of groundwater while the latter is a nonlinear diffusion of groundwater within an aquifer. Similar to saturated hydraulic conductivities in equation (26) in Kavvas et al., (2017a) for the fractional confined aquifer groundwater flow, the saturated hydraulic conductivities in equation (21) above, which governs fractional unconfined aquifer groundwater flow, are modulated by the ratios of fractional time to fractional space, $\frac{t^{1-\alpha}}{x_i^{1-\beta_{x_i}}}$, i= 1,2. In other words, the confined and unconfined groundwater diffusion in fractional time-space is modulated by the above fractional time-space ratios.

Numerical application demonstrated that as the powers of the space and time fractional derivatives decrease from 1, the recession rate of the nondimensional groundwater hydraulic heads slows down when compared to the case by the conventional governing equation (i.e., with integer order derivatives). This behavior also indicates the modulation of the nonlinear diffusion of the groundwater by the fractional powers of time and space.

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As mentioned in the Introduction section, unconfined groundwater flow is the fundamental component of the watershed runoff baseflow since it is the fundamental contributor to the network streamflow within a watershed during dry periods. As such, the behavior of unconfined groundwater flow is key to the physically-based understanding of the long memory in watershed runoff. As seen from the numerical example in Figure 3, the powers of the fractional derivatives in time and space can modulate the speed of the groundwater table evolution. Hence, they can modulate the memory of the unconfined aquifer flow, which, in turn, can modulate the memory of the watershed baseflow. Meanwhile, the Caputo derivative, as defined in its special form $D_0^{\beta_{x_i}} f(x_i)$ in space in this study, was shown by Kavvas and Ercan (2017) to be a nonlocal quantity where the effect of the boundary conditions on the groundwater flow within the flow domain can have long spatial memories with the decrease in the powers of the spatial fractional derivatives from unity. Similarly, it was shown by Kavvas et al. (2017a) that the Caputo derivative in time, $D_0^{\alpha} f(t)$, as defined in this study, is nonlocal in time, and can carry the effect of initial conditions on the groundwater flow for long times as the power in the time fractional derivative decreases from 1. Therefore, the fractional governing equation of unconfined groundwater flow in fractional time and multi-fractional space has the potential to describe the long memory characteristics of baseflow within a watershed. This important topic shall be explored in the near future.

7. **Conclusion**

A dimensionally-consistent fractional governing equation of transient unconfined aquifer groundwater flow was derived within fractional differentiation framework. After developing a fractional continuity equation, a previously-developed dimensionally consistent equation for water flux in unsaturated/saturated porous media was combined with the Dupuit approximation to obtain an equation for groundwater motion in multi-fractional space in unconfined aquifers. Combining the fractional continuity and motion equations, the governing equation of transient unconfined aquifer groundwater flow in a multi-fractional medium in fractional time was then obtained. To demonstrate the skills of the proposed fractional governing equation of unconfined aquifer groundwater flow, a numerical application was presented. As demonstrated in the numerical application results, the orders of the fractional space and time derivatives modulate the speed of groundwater table evolution, slowing the process with decrease in the powers of the fractional derivatives from 1. It is also shown that the proposed dimensionally consistent fractional governing equations approach to the corresponding conventional equations as the fractional orders of the derivatives go to 1.

Data availability.

The data used in this article can be accessed by contacting the corresponding author.

Competing interests.

The authors declare that they have no conflict of interest.

References

Atangana, A.: Drawdown in prolate spheroidal–spherical coordinates obtained via Green's function and perturbation methods, Communications in Nonlinear Science and Numerical Simulation, 19, 1259-1269, 2014.

Atangana, A. and Baleanu, D.: Modelling the advancement of the impurities and the melted oxygen concentration within the scope of fractional calculus, International Journal of Non-Linear Mechanics, 67, 278-284, 2014.

Atangana, A. and Bildik, N.: The use of fractional order derivative to predict the groundwater flow, Mathematical Problems in Engineering, 543026, 2013.

Atangana, A. and Vermeulen, P.: Analytical solutions of a space-time fractional derivative of groundwater flow equation, 381753, 2014.

Bear, J.: *Groundwater hydraulics*, McGraw, New York, 1979.

Bear, J. and Verruijt, A.: *Modeling Groundwater Flow and Pollution*, Reidel Pub. Co., Dordrecht, Holland, 1987.

Black, J. H., Barker, J. A., and Noy, D. J.: Crosshole investigations-the method, theory and analysis of crosshole sinusoidal pressure tests in fissured rock, Stripa Proj., Int. Rep, 86-03, 1986.

Cloot, A. and Botha, J.: A generalised groundwater flow equation using the concept of non-integer order derivatives, Water SA, 32, 1-7, 2006.

Diethelm K. The mean value theorems and a Nagumo-type uniqueness theorem for Caputo's fractional calculus, Fract Calc Appl Anal; doi:10.2478/s13540-012-0022-3, 15(2): 304–13, 2012.

Ercan, A. and Kavvas, M. L.: Fractional governing equations of diffusion wave and kinematic wave open-channel flow in fractional time-space. II. Numerical simulations, Journal of Hydrologic Engineering, 20, 04014097, 2014.

Ercan, A. and Kavvas, M. L.: Time–space fractional governing equations of one-dimensional unsteady open channel flow process: Numerical solution and exploration, Hydrological Processes, 31(16), 2961-2971, 2017.

Freeze, R. A. and Cherry, J. A.: *Groundwater*, Prentice-Hall, 1979.

Kang, P. K., Le Borgne, T., Dentz, M., Bour, O., and Juanes, R.: Impact of velocity correlation and distribution on transport in fractured media: Field evidence and theoretical model, Water Resources Research, 51, 940-959, 2015.

Kavvas, M.L., Kim, S., and Ercan, A.: Fractional ensemble average governing equations of transport by time-space nonstationary stochastic fractional advective velocity and fractional dispersion. I: Theory, Journal of Hydrologic Engineering, 20(2), 04014039, 2015.

Kavvas, M.L. and Ercan, A.: Time-Space Fractional Governing Equations of Unsteady Open Channel Flow, Journal of Hydrologic Engineering, 22(2), 04016052, 2016.

Kavvas and A. Ercan: Closure to discussion on ‘Time-Space Fractional Governing Equations of Unsteady Open Channel Flow’ by M.L. Kavvas and A. Ercan, J. of Hydrologic Engineering, Vol. 22, No. 9, 2017.

Kavvas, M.L., Tu, T., Ercan, A., and Polsinelli, J.: Fractional Governing Equations of Transient Groundwater Flow in Confined Aquifers with Multi-Fractional Dimensions in Fractional Time, Earth Syst. Dynam., 8, 921-929, 2017a.

Kavvas, M. L., Ercan, A., and Polsinelli, J.: Governing equations of transient soil water flow and soil water flux in multi - dimensional fractional anisotropic media and fractional time, Hydrol. Earth Syst. Sci., 21, 1547-1557, 2017b.

Khalil, R., Al Horani, M., Yousef, A., and Sababheh, M.: A new definition of fractional derivative, Journal of Computational and Applied Mathematics, 264, 65-70, 2014.

Kim, S., Kavvas, M., and Ercan, A.: Fractional ensemble average governing equations of transport by time-space nonstationary stochastic fractional advective velocity and fractional dispersion. II: Numerical investigation, Journal of Hydrologic Engineering, 20(2), 04014040, 2015.

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Earth System Dynamics Discussions

Li, M.-F., Ren, J.-R., and Zhu, T.: Series expansion in fractional calculus and fractional differential equations, arXiv preprint arXiv:0910.4819, 2009.

Li, Z. and Zhang, Y.-K.: Quantifying fractal dynamics of groundwater systems with detrended fluctuation analysis, Journal of Hydrology, 336, 139-146, 2007.

Mehdinejadiani, B., Jafari, H., Baleanu, D.: Derivation of a fractional boussinesq equation for modelling unconfined groundwater, Eur Phys J–Spec Top, 222(8):1805–12, 2013.

Metzler, R. and Klafter, J.: The random walk's guide to anomalous diffusion: a fractional dynamics approach, Physics reports, 339, 1-77, 2000.

Odibat, Z. M. and Shawagfeh, N. T.: Generalized Taylor's formula, Applied Mathematics and Computation, 186, 286-293, 2007.

Olsen, J.S., J. Mortensen, and A. S. Telyakovskiy: A two-sided fractional conservation of mass equation, *Adv. Water Resour.*, *91*, 117–121, doi: 10.1016/j.advwatres.2016.03.007; 2016.

Podlubny, I.: Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications, Academic press, 1998.

Sidle, R. C., Nilsson, B., Hansen, M., and Fredericia, J.: Spatially varying hydraulic and solute transport characteristics of a fractured till determined by field tracer tests, Funen, Denmark, Water Resources Research, 34, 2515-2527, 1998.

Sudicky, E., Cherry, J., and Frind, E.: Migration of contaminants in groundwater at a landfill: A case study: 4. A natural-gradient dispersion test, Journal of Hydrology, 63, 81-108, 1983.

Tu, T., Ercan, A., and Kavvas, M. L.: Fractal Scaling Analysis of Groundwater Dynamics in Confined Aquifers, Earth Syst. Dynam. Discuss., 8, 931-949, 2017.

Usero, D.: Fractional Taylor series for Caputo fractional derivatives. Construction of numerical schemes, URL http://www. mat. ucm. es/deptos/ma/inv/prepub/new/2007-10. pdf, 2007.

Van Tonder, G., Botha, J., Chiang, W.-H., Kunstmann, H., and Xu, Y.: Estimation of the sustainable yields of boreholes in fractured rock formations, Journal of Hydrology, 241, 70-90, 2001.

Wang, H. F. and Anderson, M. P.: *Introduction to groundwater modeling: finite difference and finite element methods*, Academic Press, 1995.

Wheatcraft, S. W. and Meerschaert, M. M.: Fractional conservation of mass, Advances in Water Resources, 31, 1377-1381, 2008.

https://doi.org/10.5194/esd-2019-37
Preprint. Discussion started: 29 July 2019

Yu, X., Ghasemizadeh, R., Padilla, I. Y., Kaeli, D., and Alshawabkeh, A.: Patterns of temporal scaling of groundwater level fluctuation, Journal of Hydrology, 536, 485-495, 2016.

Zhang, Y., Benson, D. A., and Reeves, D. M.: Time and space nonlocalities underlying fractional-derivative models: Distinction and literature review of field applications, Advances in Water Resources, 32, 561-581, 2009.

https://doi.org/10.5194/esd-2019-37
Preprint. Discussion started: 29 July 2019

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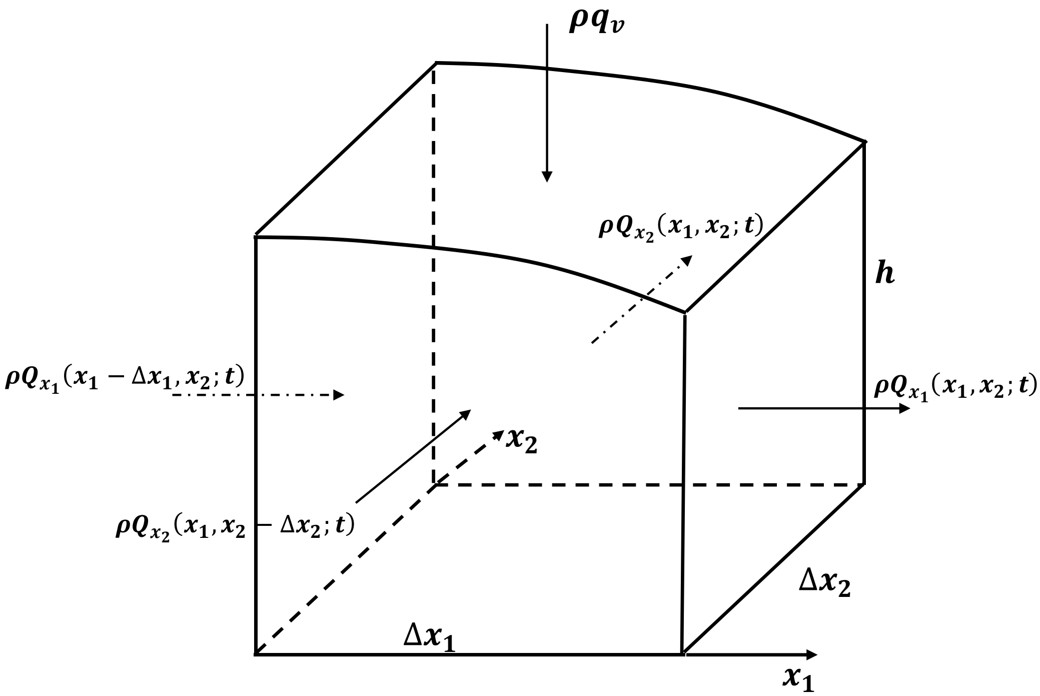

Figure 1. The mass flux through the control volume of an unconfined aquifer.

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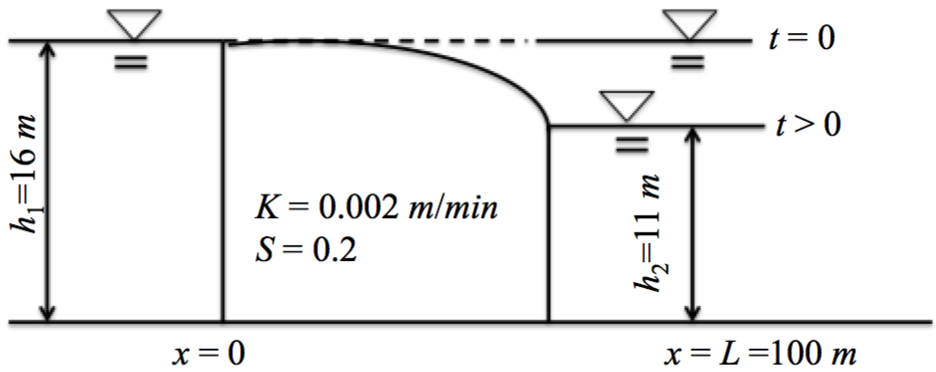

Figure 2. The sketch of the problem of the water seepage through a dam's body as an unconfined groundwater flow

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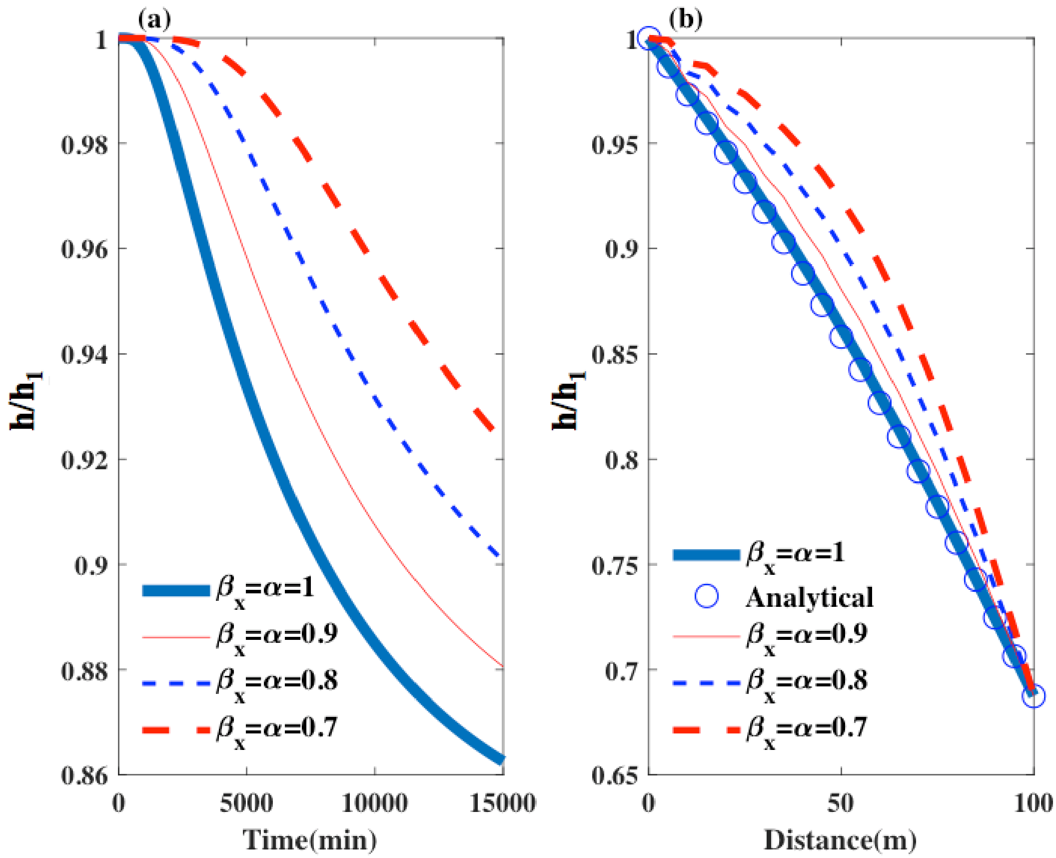

Figure 3. (a) The normalized groundwater head h/h_1 at $x=L/2$ through time under different fractional derivative powers; (b) The normalized groundwater head h/h_1 at t=15000 min through length of the aquifer (through the body of the dam) and the analytical solution of the standard governing equation of unconfined groundwater flow when $\beta_x = \alpha = 1$ at the steady state.