Response to Comments of Anonymous Referee #1

Authors thank Reviewer #1 for the valuable comments and suggestions, which helped to improve the manuscript considerably. Our responses are in blue color below.

Referee #1: General comments:
This paper deals with the theoretical study of deriving the governing equation of unconfined aquifer flow using Caputo fractional derivative approach. The derivation process is very clearly presented for the reader to understand. Including further discussion of the following is expected to further enhance the value of the fundamental research work.
Authors' response: Thank you, please see our responses below.

Referee #1: Specific comments: 1) The paper needs to contain the minimum information of the numerical scheme needed to draw Figure 3. This will provide important information to persuade the paper's reproducibility.
Authors' response: Information about numerical scheme is added as an appendix to the revised manuscript.

Referee #1: 2) The authors simulated a state after a very long time to draw Figure 3 (b). For integer cases, one can derive a simple steady-state analytical solution, as shown in Eq. 24. However, This reviewer is curious about what the fractional case might look like. It is necessary to include the authors' views on this curiosity.
Authors' response: Derivation of an analytical solution for the proposed fractional governing equation of unconfined groundwater flow is not the focus of this study. Since such an analytical solution is not available currently, the authors will address this issue in future studies.

Referee #1: 3) Further discussion is needed about the time required to converge to a steady state. The time required will naturally be affected by the fractional order.
Authors' response: Further discussion is provided in the revised manuscript for two numerical applications for various fractional orders.

Referee #1: 4) In addition to the head results, the authors need to explain the behavior of the discharge. In the case of integer cases, the discharge at steady-state can be derived analytically simply, but what happens in the case of fractional cases, and the effects of fractional order on steady-state discharge need to be discussed further.
Authors' response: A figure for the discharge is added to the revised manuscript (Figure 3c, see below). A discussion of the new figure for flows by the standard integer order and fractional order governing equations are also provided in the revised manuscript. It takes longer time to achieve steady state conditions as fractional powers decrease from 1 toward zero.

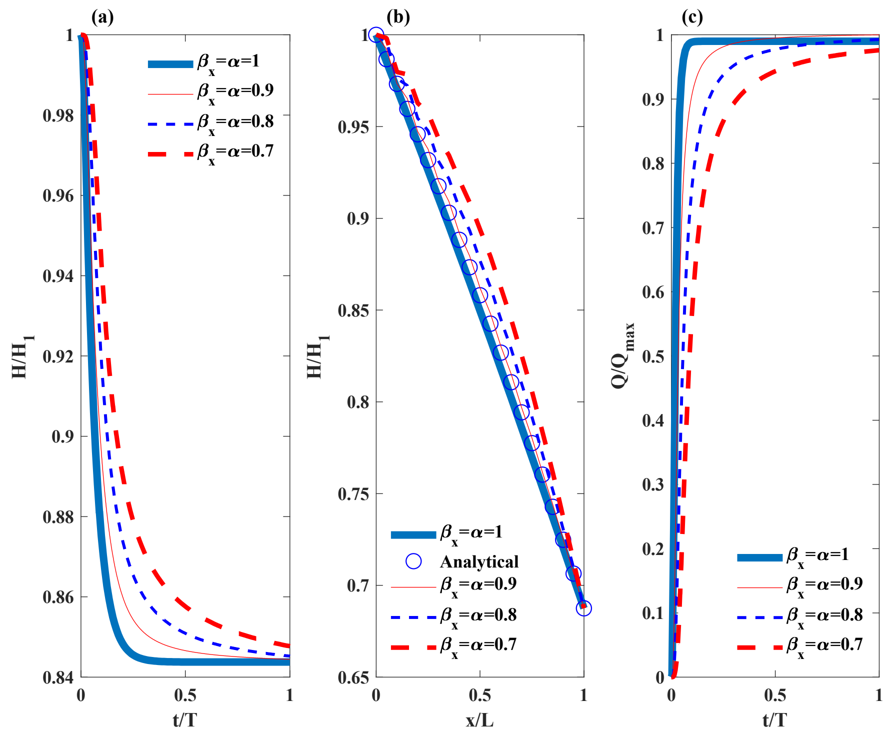

Figure 3. Results for numerical application 1: (a) The normalized groundwater head h/h_1 at $x=L/2$ through time under different fractional derivative powers; (b) The normalized groundwater head h/h_1 at t=40,000 min through the length of the aquifer (through the body of the dam) and the analytical solution of the conventional governing equation of unconfined groundwater flow when $\beta_x = \alpha = 1$ at the steady state; (c) The normalized groundwater discharge per unit width at $x=L/2$ through time under different fractional derivative powers; t is time and the simulation time T is 120,000 min.

In order to further satisfy reviewer's comment on flows, we added a second numerical example. The second problem deals with a transient unconfined groundwater flow from a hillslope toward a stream (Figure 4, see below). The upstream boundary vertical plane separates the region of flow from the adjacent hillslope that feeds the adjacent tributary system, therefore $\frac{\partial h}{\partial x} = 0$ at x=0 (Freeze, 1978).

As shown in Figure 5c in the revised manuscript (see below), the newly-developed governing equations can produce heavy-tailed recession behavior in unconfined aquifer discharges by changing fractional powers.

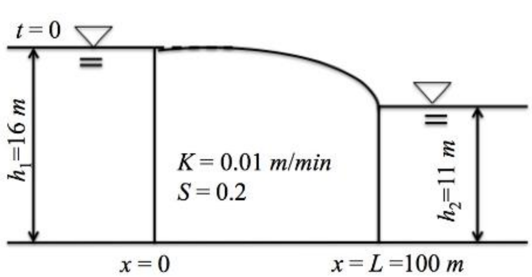

Figure 4. The sketch of numerical application 2: The downstream groundwater head is fixed at 11 m and the initial upstream groundwater head is 16 m.

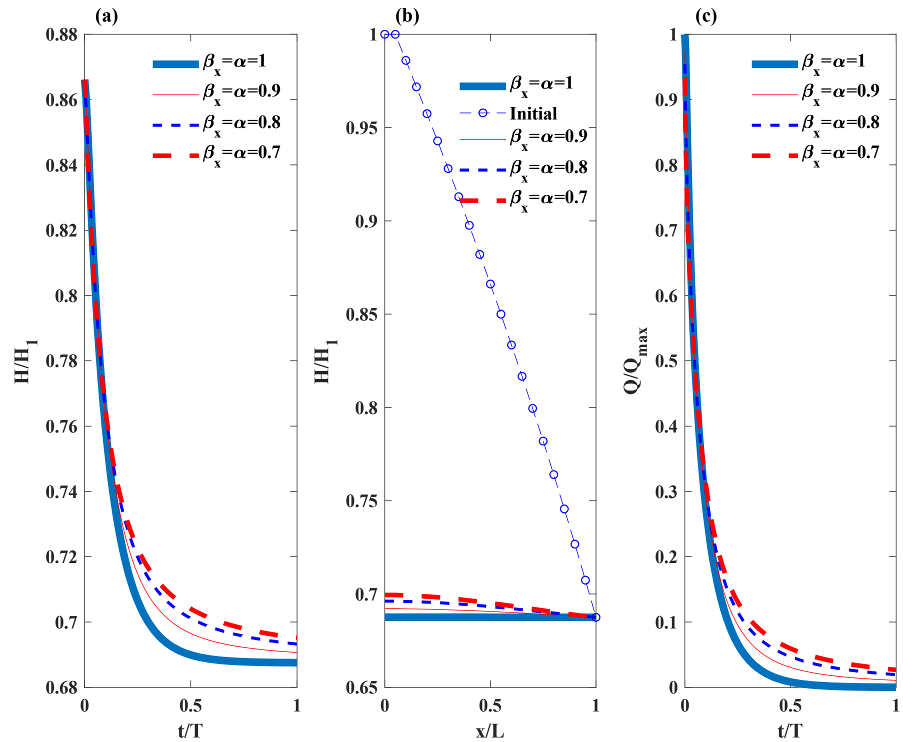

Figure 5. Results for numerical application 2: (a) The normalized groundwater head H/H_1 at $x=L/2$ through time under different fractional derivative powers; (b) The normalized initial groundwater head in the unconfined aquifer, and the normalized groundwater head H/H_1 at time t=60,000 min through length of the aquifer under different fractional derivative powers; (c) The normalized groundwater discharge per unit width at $x=L/2$ through time under different fractional derivative powers; t is time and the simulation time T is 60,000 min.

Reference

Freeze, R. A., “Mathematical models of hillslope hydrology”, Chap. 6 in *Hillslope Hydrology*, ed. by Kirkby, M.J. John Wiley & Sons, Ltd, New York, 1978.

Referee #1: Technical corrections:
Too many terms are given in eq. (15). Matching the order and number of terms in eqs. (14) and (15) will help readers better understand.
Authors’ response: The manuscript was revised according to the specific suggestion of the reviewer.

Referee #1: Technical corrections:
Line 255: storage coefficient S = 0.2 popped out abruptly without any explanation, and the effective porosity Sy is missing a description of what value is given in the numerical analysis.
Authors’ response: The manuscript was revised according to the specific suggestions of the reviewer.