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Reply to the Lovejoy and Varotsos comment entitled "Trained eye deceived by fractal clustering"

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Abstract. This comment from L&V contains no substantiated arguments which invalidate anything in our comment article or in our first reply. It is a lengthy collection of unsubstantiated and erroneous claims which obscures the real issue, which is:

Have L&V presented valid tests which prove that the temperature response in the climate models is inconsistent with a linear response model?

L&V are in denial about the most obvious facts. One of these is that adding a noise to a intermittent signal will reduce the intermittency. The claim of theirs that is most relevant to the linearity issue is that the statistical uncertainties are so large that they overshadow the intermittency-reducing effect of curved structure-function plots and internal variability. This assertion is unsubstantiated, false, and bizarre. If statistical errors were this important they would invalidate all L&V's results from their original paper, and we would have used it against them. *Moreover, in our previous reply, a code was made available for L&V to check this for themselves, so they have no reason for making such a claim.*

In the discussion of how to define multifractality in stochastic processes L&V confuse the concept of a Lévy *process* with that of a Lévy *flight*, and disregard the definition coined by Mandelbrot, Calvet and Fisher (MCV) in 1997. MCV define a multifractal time series as one with power-law structure functions, which implies temporal dependence in the data. L&V associate multifractality with fractal properties of the *image set* $\{X(t)|t=1,2,\ldots\}$ of the time series, with the result that temporal dependence (clustering in time) will not be necessary property of a multifractal. For instance, with the L&V definition, random shuffling of a time series will not change the multifractal properties. The trace moments used by L&V to estimate multifractal intermittency *do* change with shuffling, hence trace moments measure something else than multifractality.

All these paradoxes are resolved by defining a multifractal time series as one with power-law structure functions. With this definition one has to accept that Lévy processes (non-Gaussian white noise) are not multifractals, and that moment-based estimators yield curved scaling functions whose curvature depends on a subjectively chosen scale range for fitting a straight line to curved trace moments (spurious multifractality).

L&V claim that we "misunderstand" multifractality and trace moment analysis. We respond by elaborating on our understanding of multifractality in an appendix. In the context of linearity testing, however, our understanding of the more arcane aspects is largely irrelevant. We know how the trace moment routine works and we can test the effect various properties in the data have on the intermittency estimates. Since trace moments is the only intermittency estimator applied by L&V our conclusions are valid even if we treat it as a "black box."

Below, we respond to L&V's reply in a chronological manner, to make sure that everything is addressed.

1 Reply to "Summary"

L&V: "...we quantified something - we thought - quite straightforward, the fact that the response of the atmosphere to volcanic forcing is nonlinear."

R&R: In our attached Figure 1a we show the Gao volcanic forcing and the global temperature responses in the NorESM model. It is not at all clear that strong volcanic forcing spikes give weaker responses than weak forcing spikes as claimed by L&V. The instantaneous responses seem quite proportional to the strength of the forcing.

L&V: "The basic fact that a linear transfer function (Green's function can only make a linear modification to the structure function exponent $\xi(q)$ has been known for some time and is even not contested by R&R."

R&R: Wrong! We *d*o contest that assertion, because it is only valid under conditions I-III in our comment. L&V fail to state the conditions of validity. We proved in Sect. 2.4 in our comment that it is true only under these conditions which have been spelled out very clearly in our original comment and in our first reply, but L&V continue to ignore them. We repeat: the statement holds only if
(I): The Green's function is a power-law $G(t) \sim \Delta t^{\beta/2-1}$ on all interesting scales.
(II) The exponent $\xi(q)$ exist, i.e., if the structure functions actually *are* power laws on all interesting scales.
(III) The "response signal" does not contain a component from internal noise which will influence the intermittency estimates.

Condition II and III are the most important here. In our attached Figure 2 and 3 we demonstrate that deviations from power-law scaling in the structure function (condition II) makes the intermittency estimates depend on the particular scale range used for fitting a straight line to a curved graph. These figures are taken from a recent paper in ESD which was revived by Shaun Lovejoy. The breakdown of condition III is clearly illustrated in attached Figure 1. Figure 1b is a signal composed of two components. One is the volcanic forcing signal (red in Figure 1a) normalised such that the magnitude of the large volcanic spikes roughly match those of the volcanic response signal in NorESM (grey in Figure 1a). This signal can be thought of as the instantaneous response to the stochastic forcing. The other component is the internal variability represented by a control run. This composite signal represents a trivial linear transformation (multiplication by a normalization factor) plus a signal representative for the internal variability. Figure 1c,d shows the structure functions (SFs) and the scaling function for the Gao volcanic forcing computed from straight lines fitted to the SFs in the range displayed in Figure 1c. According to the assertion of L&V (who believe condition III is irrelevant), the intermittency shown by the curvature of the scaling function in Figure 1d should be preserved in the scaling function for the composite signal shown in Figure 1g, but it is not. The latter signal is almost non-intermittent due to the "contamination" from the internal noise. This contamination explains the reduced intermittency observed in the the response to the volcano forcing shown in Figure 1e,d. This proves that:

> *nonlinearity in the response is not required to explain the difference in intermittency between forcing and response.*

L&V: The key limitation of the analysis was the existence of a single time series for each, and these were over finite ranges of time scales... These are the true limitations of our analysis and conclusions. R&R's hypotheses I-III are thus irrelevant as indicated in our response.

R&R: In Figure 1 we showed that condition III alone is sufficient to explain the entire difference in intermittency between forcing and observed response. The same was shown in our linear oscillator example in our comment article. In their efforts to escape from this conclusion, the authors now claim that the errors due to finite sample size are so large that they overshadow this intermittency-reducing effect of internal variability, and render this effect irrelevant. But if these errors were this large they necessarily would make the observed difference in intermittency statistically insignificant. A code has been made available for L&V to check that the statistical uncertainty is small. Hence,

> *the finite-sample error argument L&V is false and, if it were true, would invalidate their own conclusions.*

2 Response to "The untrained eye works well"

We stressed in our previous reply that the "trained eye" of course isn't a substitute for data analysis. But it is a fundamental principle in data analysis to perform a careful inspection of the data. For instance, the inspection of the signals of volcanic forcing and response put together in attached Figure 1a provides important information about the linearity in the instantaneous response to volcanic spikes that is not so easily quantified by analysis.

In the second paragraph on page 2, L&V suggest that we interpret "clustering" as synonymous with "fractal" (non-integer fractal dimension). This is a misunderstanding. We never expressed that idea. The discussion on pages 3 and 4, and Figs. 2-5, is based on two fundamental misconceptions:

1. L&V do not distinguish between a *Lévy process* and a *Lévy flight*. They write: "Recall that Levy processes have long power law tails on their probability distributions." This is incorrect. Lévy processes is a broad class of processes with independent increments. The

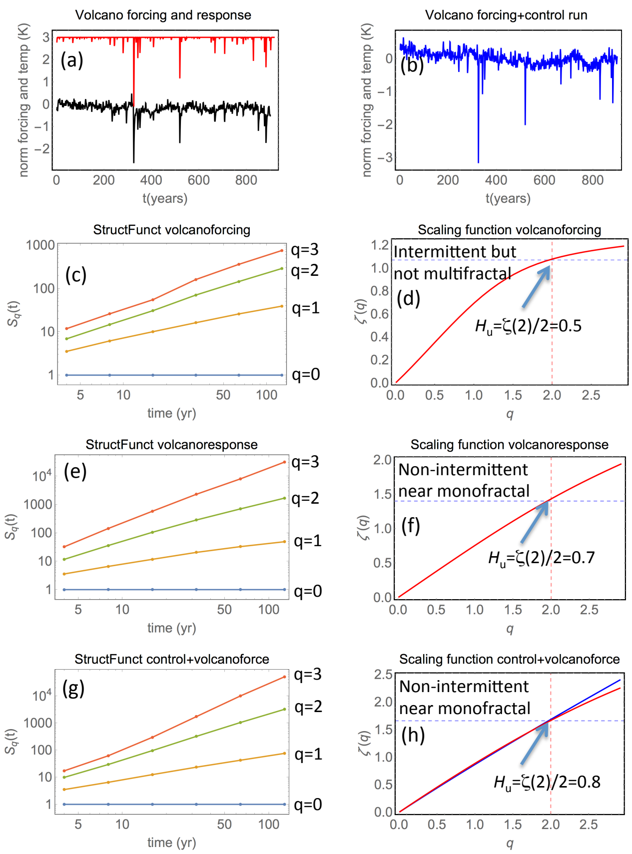

Figure 1. Analysis of global temperature responses in the NorESM model. (a): the Gao volcanic forcing (red) normalized such that the larges spikes are approximately equal to the spikes of the response signal (black). (b): the red curve in (a)+the control-run temperature signal. (c): structure functions (of cumulative sum) of volcano forcing. (d): scaling function derived from (c). (e): structure functions of the volcanic response. (f) scaling functions derived from (e). (g): structure functions of the signal in (b). (h): scaling function derived from (g). The red line arise from fitting straight lines in the entire scale range plotted, 4-128 yr. The blue line is from fitting only in the scale range 16-128 yr. It shows weak intermittency in both cases, but also that estimated intermittency depends on the scale range chosen for fitting. *The difference in curvature (reduction of intermittency) between (b) and (h) is exclusively caused by the addition of the internal noise represented by the control run.*

theory of Lévy processes was developed in the 1920s and 1930s, and we advice L&V to take a look at the excellent review of Appelbaum (2004). L&V are thinking of *Lévy flights*, which is just a small subclass of the Lévy processes where the PDF is so heavy-tailed that the variance is infinite. There is no reason to assume that any of the processes under discussion here are that heavy-tailed.

2. Their discussion revolves around clustering in the *image set* of the random variable; $\{X(t)\}|t=,1,2\ldots\}$ for a Lévy flight, while we are discussing the clustering of spikes *in time* of spikes created by a *Lévy noise* (which is the increments of a Lévy process). The image set contains no information of the timing of the fluctuations, so shuffling of the data in time makes no change in the image set.

If the image set $\{X(t)\}|t=,1,2\ldots\}$ is what defines multifractality we face disturbing implications. One is that dependence in the data will be irrelevant. Such dependence (clustering of spikes in time) is the most prominent feature of a multiplicative cascade construction (e.g., a β-model). If we generate such a multifractal, and then perform a random shuffling of the data in time, we convert the data into realisations of a Lévy process. As we understand L&V, they consider the shuffled time series to be realisations of the same multifractal as obtained from the β-model. But we could also construct realisations of this Lévy process by drawing random numbers from the same non-Gaussian PDF (a Lévy-noise construction). Hence, if L&V are right the time series generated from the two very different constructions should possess the same multifractal intermittency. This is discussed in detail in the appendices of Rypdal and Rypdal (2016),[1] where Lovejoy was a very active referee. We find it strange that L&V do not refer to this paper and the associated discussion. There, we proved analytically that shuffling creates curved structure functions in a log-log plot, and in attached Figure 3 we demonstrate this by performing the constructions numerically. From this figure we observe that the structure functions and trace moments are different in two constructions which according to L&V represent the same multifractal. Hence, *we have to conclude that the trace moment analysis does not detect correctly the multifractality*.

The root of this paradox his the association of multifractality with the image set. The abstract geometric definition of multifractals is useful for theoretical purposes, but for estimation from data we have to resort to the moment-based approaches (the link between the two is the Legendre transform, but is subtle when we deal with finite data sets. We discuss this in some detail in in the appendix). This moment-based approach is applied by Mandelbrot, Fischer, and Calvet (1997)[2], where a multifractal is defined as one with power-law structure functions (a more correct term for the structure-function estimates is *empirical moments*[4]). This is a reasonable definition, because the scaling function and the intermittency parameters can only be estimated in a unique way when the structure function are straight lines in a log-log plot.

[1]M. Rypdal and K. Rypdal, Late Quaternary temperature variability described as abrupt transitions on a 1/f noise background, Earth Syst. Dynam., 7, 281-293, 2016, doi: 10.5294/esd-7-281-2016. http://www.earth-syst-dynam.net/7/281/2016/esd-7-281-2016.pdf

[2]B. Mandelbrot, A. Fischer, and L. Calvet,
A Multifractal Model of Asset Returns, Cowles Foundation Discussion Paper # 1164, September 15, 1997,
http://users.math.yale.edu/ bbm3/web_pdfs/Cowles1164.pdf

If they are curved these estimates will depend on the chosen fitting range. With this definition it can be proven that multifractal intermittency is associated with temporal dependence in the data.[1] Loosely speaking, multifractality implies that spikes are not randomly distributed in time, but are clustered in groups along the time axis.

Attached Figures 2 and 3b demonstrate that estimates from structure functions and trace moments yield considerable intermittency for non-Gaussian Lévy processes. According to the definition of Mandelbrot et al., these processes are not multifractals. This is the justification of our statement; *"trace moment analysis only detects non-Gaussianity, not multifractal clustering."* This answers L&V's "Minor comments, point 3."

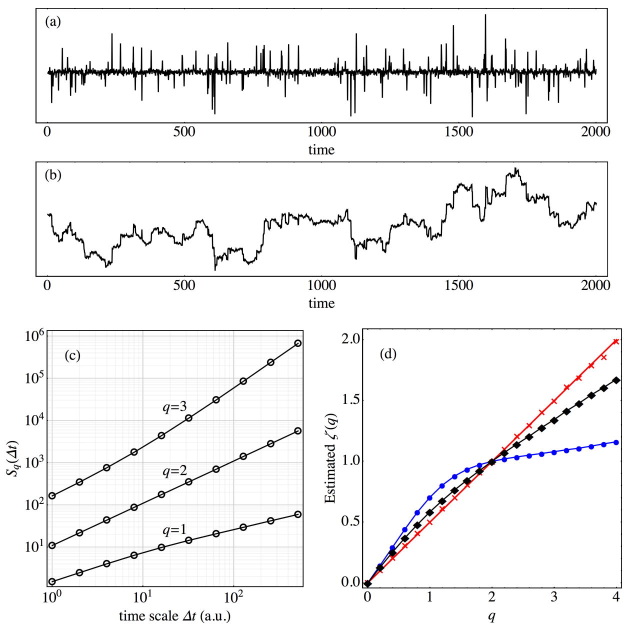

Figure 2. (a): The increments of a jump-diffusion process shown in (b). This is a non-Gaussian independent noise process. (b): A realisation of a jump-diffusion process, and the cumulative sum of the signal in (a). This process is the sum of a Brownian motion and a Poisson jump process as described in Appendix B of Rypdal and Rypdal (2016).[1] The jump distribution is Gaussian with a standard deviation that is ten times greater than the standard deviation of the increments of the Brownian motion. (c): $S_q(\Delta t$ for $q = 1, 2, 3$ for the jump-diffusion process as computed from a large ensemble of realisations of the process. (d): Scaling function $\zeta(q)$ estimated from structure functions like those in (c). The red line is estimated by computing the slope of the structure-function curves on the longest time scale ($\Delta t = 500$). The blue curve is estimated from the slopes at the shortest time scale ($\Delta t = 1$). The black curve by estimating the slope of the straight line drawn between the end points of the structure-function curves. *This demonstrates that for processes with structure functions that are not power laws, and according to the MFC definition are not multifractals, the estimated intermittency depends on the scaling regime chosen for fitting.*

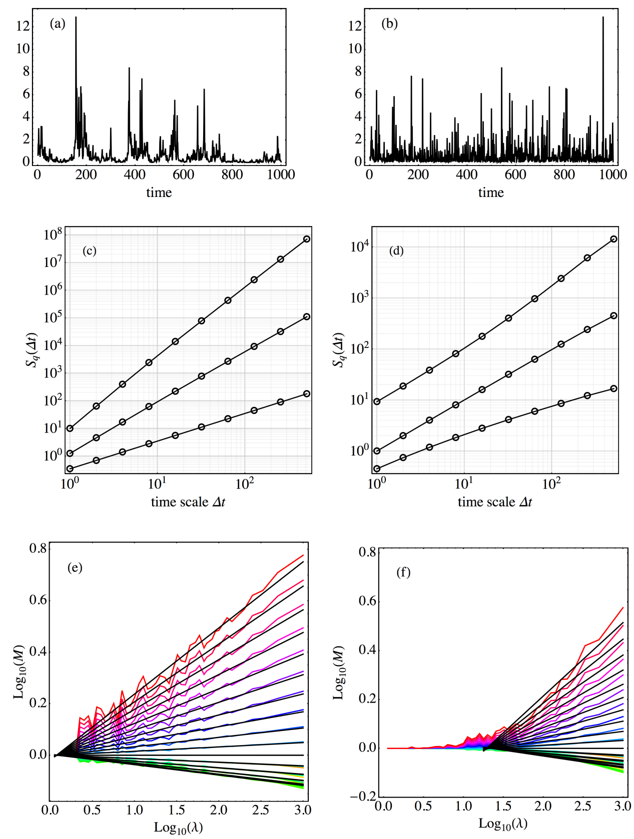

Figure 3. (a): realisation of a multiplicative cascade process. (b): the same process after random shuffling of the data. These two time series have the same image set, and according to L&V they have the same multifractal properties. (c) and (e): structure functions and trace moments of the realisation in (a). (d) and (f): the same for the realisation in (b). *Note the curvature of the plots in (d) and (f) for $q \neq 2$. The straight plot for $q = 2$ is a consequence of the independence in the time series in (b). The curvature of the plots influences the intermittency estimates and creates "spurious multifractality."*

Figure 5 in the reply of Lovejoy and Varotsos show that estimated scaling is similar for a Lévy process and a multifractal process. This is well known. See for instance the work of Neumann (2010)[3] or Heyde and Sly (2008).[4] This is called spurious multifractality; standard estimators may lead the scientist to conclude that a process has the characteristic of a multifractal, when in fact it does not.

> *Although the definition of multifractality is interesting to some, and important in some contexts in Earth system dynamics, it is largely irrelevant to the conclusions in the paper by L&V. All conclu-*

[3] Neumann S.: Apparent/spurious multifractality of data sampled from fractional Brownian/ Lévy motions, *Hydrol. Process.*, 24, 2056-2067, 2010.

[4] Heyde, C., and Sly, A.: A cautionary note on modeling with fractional Lévy flights, Earth. Physica A., 387, 5024-5032, 2014.

> *sions on intermittency there are drawn from the trace moment analysis, and what's important is which properties of the signal this analysis measures, and not how we define multifractality.*

3 Reply to "minor comments"

3.1 The statistical test argument

The arguments and analogies L&V present here are so bizarre that we will not try to argue against them. We shall just state our point of view, which we believe is mainstream in the philosophy of science.[5]

We agree with L&V that "it is impossible in principle to prove linearity from data or from numerics." In fact, it is commonly accepted that *no falsifiable statement about nature can be proven or verified*, since there is always a chance that a new observation or a new test may prove it false. On the other hand, *only one single observation or test may be sufficient to falsify a well-posed hypothesis*. Hence, we cannot prove linearity, but we can falsify it, because from the linearity assumption we can make predictions that can be tested against observation. If we can falsify that the response is linear, i.e., if we can demonstrate that observation is inconsistent with a linear response, then we have proven that the response his nonlinear. We can do this because nonlinearity is a negation of a falsifiable statement. For the same reason it is impossible to falsify nonlinearity. The strongest conclusion we can draw is that the observations and tests at hand so far have failed to reject the linearity hypothesis. It should also be said, however, that if a broad range of tests fail to falsify linearity, then we will gain strong confidence in the hypothesis that the response is linear (in the sense that nonlinearity is so weak that it cannot be detected). This is the law of induction.

> *L&V may not be aware of it, but the two tests they have employed in their paper are exactly of this type. Our criticism is not of the logic of this test, but of the way it was done, and how the results were presented.*

3.2 Reply to "Linear oscillators"

In our comment we demonstrate that the trace moments of a particular linear response to volcanic + stochastic forcing (using the Green's function of a damped harmonic oscillator) is very similar to those of the Zebiak Cane model, demonstrating that a linear response plus internal variability can give a low-intermittency output from a high-intermittency input. What L&V insinuate is that we have selected a realisation that looks like ZC-output and in this way obtained trace moments that accidentally are similar to that of the ZC-output. Maybe this is L&V's way of doing science, but it is not ours. The oscillator model output and the trace moment output are very similar in different realisations, and the ensemble mean of the trace moments will be smoother but have the same structure and give very similar intermittency parameters. In our previous reply we gave a URL where L&V can download a Mathematica notebook with the necessary routines to check this for themselves.

The point with our demonstration, however, was not to produce a response that imitates the ZC-model, but to demonstrate that a linear model with a reasonable level of internal variability can provide a strong reduction of the output intermittency, contrary to the claims of L&V. Another demonstration of this is shown in attached Figure 1b,d,f as discussed previously in this reply. Here the linear filter was simply multiplication by a normalization factor, and the reduction of intermittency was exclusively caused by the internal variability component (condition III).

The acrobatics L&V perform to escape from the conclusion that follows from these examples are similar to, and equally ungraceful, as their finite-sample error argument in "Summary." And it is again a boomerang on themselves. If the ensemble uncertainty of the trace moments is so large that the reduction of intermittency we observe is accidental it could be so also in the the actual ZC-data.

3.3 Reply to "Misunderstanding the trace moment analysis"

This point has been responded to in section 2 of this reply.

3.4 Reply to "point 4"

It is hard to take L&V's complaint about our notation seriously. In this cross-disciplinary field there is *absolutely no standard notation*. Much of he notation used in Lovejoy's writing is quite alien to us, and is closely connected to his own methodology. Often we have reservations against these methods, so why should we adopt his particular notation? It is part of our profession to accept other notations, as long as they are properly defined, as it was in our case.

Then to the particular points: In turbulence the fluctuations of the velocity field is growing with spatial scale. Translated to time series this corresponds to *motions* ($H > 0$). This is why structure functions in turbulence are computed directly from the increments of the velocity field itself. In geophysical time series a common situation is that the signal is a *noise* ($H < 0$). Structure functions computed directly from a noise provide no useful information. The standard procedure is therefore to create a motion by forming a cumulative sum, and to form the structure function on this signal. It is not particularly logical to change the notation S_2 because of this procedure.

The last sentence under this point reads: "In any case…there is no "curved scaling function" to be "incorrectly interpreted by L&V" (compare Fig. 2 and 3 of L&Vr)."

[5] Bird, A., Philosphy of Science, Routledge, 1998.

This statement demonstrates again that L&V still don't understand how structure functions work. In L&Vr Fig. 2 the first and second order structure functions are computed directly from the noisy signal. These structure functions are completely flat (constant) for *any* signal for which fluctuations do not grow with scale. They are completely insensitive to those properties that create the curvature in the the structure functions of the cumulative sum. In Fig. 3 L&V plot only the square root of the second-order Haar structure function. The $q=2$ structure function is the only structure function that is not curved in an uncorrelated noise. This was a major issue in the discussion with Lovejoy in ESDD associated with the recently published paper,[1] where a rigorous proof was given. It is also demonstrated in attached Figure 2.

4 Reply to "Conclusion"

In the first pararagraph of the conclusion L&V reiterate their confusion concerning Lévy processes vs. Lévy flights. Non-Gaussian Lévy processes, that do not possess the extremely heavy tails of Lévy flights, do *not* exhibit characteristics that are indistinguishable from time series constructed from multiplicative cascades. This is clearly demonstrated in our Figure 3, where the structure functions and trace moments are changed from straight lines to curves in log-log plots. The problem is that mindless use of these moment-based estimators to construct scaling functions and estimate intermittency parameters often give similar results. The statistics and structure functions do *not* "show that quite different multifractal production mechanisms can lead to very similar statistics." The statistics is different, the structure functions/trace moments are different, it is just the intermittency estimated from these that sometimes give similar results. Such estimates from non-power law moments are not unique and have no real meaning.

The second paragraph seems to reiterate that L&V consider statistical errors (due to finite sample size and only one realisation available for analysis) are so large that they render our conclusions about the effect of curved structure functions and internal noise invalid (statistical insignificant). Here they admit that this argument may boomerang on their own results, but for some unspecified reason they conclude that "there are no compelling arguments to doubt our conclusions." The fact is that the statistical error limitation applies in principle to L&V's tests, because they are limited to finite sample size and one single realisation. But simple error estimates indicate that this is not a serious limitation for these data. Our test with the linear oscillator model, however, does *not* have this limitation. We can run as many realisations as we want, and we have of course checked that statistical errors are insignificant and do not invalidate our conclusions. If L&V still contend that we are cheaters they should run our Mathematica routine and prove it.

Appendix A: Multifractals and Lévy processes

A1 A definition based on the qth moments

Mandelbrot et al.[2] give the following definition of a multifractal stochastic process:

Definition. *Let $\stackrel{d}{=}$ denote equality in distribution. A stochastic process $X(t)$ with stationary increments is multifractal if*

$$X(t+a\Delta t)-X(t)\stackrel{d}{=}M(a)\Big(X(t+\Delta t)-X(t)\Big),$$

where $M(a)\geq 0$ is a family of random variables satisfying the relation $M(ab)\stackrel{d}{=}M_1(a)M_2(b)$ with M_1 and M_2 being independent realizations of M.

A self-similar process $X(t)$ is a special case of a multifractal process with $M(a)=a^h$ being a deterministic function of the scale a. For a multifractal process we have

$$\langle M(ab)^q\rangle=\langle M(a)^q\rangle\langle M(b)^q\rangle,$$

which implies that $\langle M(a)^q\rangle$ is a power law $\langle M(a)^q\rangle=a^{\zeta(q)}$. This implies the scaling relation

$$\begin{aligned}S_q(\Delta t)&=\langle|X(t+\Delta t)-X(t)|^q\rangle\\&=\langle M(\Delta t)^q\rangle\langle|X(t+1)-X(t)|^q\rangle\\&=c_q\Delta t^{\zeta(q)},\end{aligned}$$

where $c_q=\langle|X(t+1)-X(t)|^q\rangle$.

The functions $S_q(\Delta t)$ are called the structure functions, and most methods for estimating multifractality in observational data are based on structure functions, or on closely related constructions such as wavelet-based fluctuation functions. We stress that from the definition of Mandelbrot *et al.*, a multifractal process must have structure functions that are power-laws in the scale Δt, and only if this is satisfied is the scaling function $\zeta(q)$ defined.

Let us now recall the definition of a Lévy process.

Definition. *A stochastic process $X(t)$ is called a Lévy process if $X(0)=0$ almost surly and*

1. *Increments are independent and stationary.*
2. *$X(t)$ is stochastically continuous with càdlàg paths.*

The second condition is a technical requirement that is not important for the considerations here, and one should think of a Lévy process as a continuous time random walk where the increments are not necessarily Gaussian. The simplest case of a Lévy process is the Wiener process (or Brownian motion) for which the increments $X(t+\Delta t)-X(t)$ *are* Gaussian. This process is self-similar with a self-similarity exponent $h=1/2$, i.e. we have a linear scaling function $\zeta(q)=q/2$. Another class of self-similar Lévy processes are the so-called Lévy flights for which the increments have α-stable (heavy-tailed) distributions. For Lévy flights with stability parameter $\alpha < 2$ we have monofractal scaling $\zeta(q) = q/\alpha$ for $q < \alpha$, and the scaling function is not defined for $q > \alpha$ due to the heavy tails. However, the self-similarity relation $X(at) = a^h X(t)$ is valid with $h = 1/\alpha$. It is well known that if one attempts to estimate the scaling function for a Lévy flight using structure functions, then spurious multifractality is observed (Heyde and Sly, 2008). The extreme tails in the increment distribution of a Lévy flight are often unrealistic models of real world observations, and it is usually sufficient to consider non-Gaussian distributions for which all the moments exist. One can for instance use truncated Lévy flights[6], or any other non-Gaussian Lévy process.

Proposition. *A Lévy process with finite moments is not a multifractal stochastic process in the sense of Mandelbrot et al., unless it is the Wiener process.*

The proof is simply to show that Lévy processes that are non-Gaussian have structure function that are not power laws. This is done in the appendies of Rypdal and Rypdal (2016)[1]. For the purpose of modelling volcano forcing as a stochastic process (for instance as the increments of a multifractal process or as the increments of a Lévy process) we can draw the following conclusion:

> *If the statistical properties of the volcano forcing signal are invariant under shuffling, then it is not consistent with a multifractal stochastic process in the sense of Mandelbrot et al.*

A2 A definition based on the singularity spectrum

An alternative way of defining what is meant by a multifractal stochastic process is via the so-called singularity spectrum. For a realization of the stochastic process $X(t)$ one can define the local Hölder exponents as

$$\gamma(t) = \liminf_{\substack{s \to t \\ s \neq t}} \frac{\log|X(s) - X(t)|}{\log|s-t|},$$

with the convention that $\log 0 = -\infty$. The definition ensures that $\gamma(t)$ exists for any realization of $X(t)$ for all t, and if we have asymptotic scaling

$$|X(t+\Delta t) - X(t)| \sim |\Delta t|^{\gamma'} \text{ as } \Delta t \to 0,$$

then $\gamma(t) = \gamma'$. For some stochastic processes there is essentially only one Hölder exponent γ. For instance, a realisation of a Wiener process has with probability one $\gamma(t) = 1/2$ for all time instances t.

The singularity spectrum for a realisation of a stochastic process is a function $f(\gamma)$ that specifies the fractal dimensions of the level sets of the function $\gamma(t)$:

$$f(\gamma) = \dim_H K_\gamma,$$

where $K_\gamma = \{t \in \mathbb{R} : \gamma(t) = \gamma\}$. Here $\dim_H(\cdot)$ denotes the Hausdorff dimension, and the convention is that the dimension of the empty set $\emptyset$ is $-\infty$. For a Wiener process we have $K_{1/2} = \mathbb{R}$ and $K_\gamma = \emptyset$ for $\gamma \neq 1/2$, and hence the singularity spectrum is

$$f(\gamma) = \begin{cases} 1 & \gamma = 1/2 \\ -\infty & \text{else} \end{cases}.$$

Since $f(\gamma)$ has a positive value for only one particular Hölder exponent, the Wiener process is often termed monofractal.

If a process is a multifractal in the sense of Mandelbrot *et al.*, so that its scaling function $\zeta(q)$ is defined, then one can in some cases express the relationship between the scaling function and the singularity spectrum via the Legendre transform:

$$f(\gamma) = \inf_q \{q\gamma - \zeta(q) + 1\}. \tag{A1}$$

If this relation holds, then a monofractal singularity spectrum corresponds to a linear scaling function, and hence to a self-similar process. And if the scaling function $\zeta(q)$ is strictly concave, then this corresponds to a non-trivial singularity spectrum where there is a range of γ-values for which the level sets K_γ have positive Hausdorff dimensions. In other cases, such as for α-stable Lévy processes (so-called Lévy flights), the relation in Eq. (A1) does not hold since $\zeta(q)$ is not defined for $q > \alpha$. However, it has been shown by Heyde and Sly (2008) that the estimated (using standard methods) scaling function of a Lévy flight is

$$\hat{\zeta}(q) = \begin{cases} q/\alpha & q < \alpha \\ 1 & q > \alpha \end{cases}, \tag{A2}$$

and it has been shown by Jaffard (1999)[7] that the singularity spectrum has the form

$$f(\gamma) = \begin{cases} \alpha\gamma & q \in [0, 1/\alpha] \\ -\infty & \text{else} \end{cases},$$

which is the Legendre transform of the expression in Eq. (A2). In this sense, one might say that the mulitfractal formalism works for α-stable Lévy processes. On the other hand, there are heavy-tailed Lévy processes that have monofractal singularity spectra, even though the estimated scaling functions are concave. Hence, one should be careful to interpret a signal as multifractal based on the estimated scaling function alone. In the next subsection we will list some results that show that a reasonable Lévy-process description of the volcanic forcing signal must have a monofractal singularity spectrum.

[6] Terdik, G., Woyczynski, W. A., and Piryatinska, A.: Fractional- and integer-ordered moments, and multiscaling of smoothly truncated Lévy flights. Phys. Lett. A, 348, 94-109, 2006.

[7] Jaffard, S.: The multifractal nature of Lévy processes. Probability Theory and Related Fields ,114, 207-227, 1999.

A3 The singularity spectra of Lévy processes

A Lévy process $X(t)$ is conveniently characterized by the characteristic functions of the variables $X(t)$. These are defined as $\phi_{X(t)}(u) = \langle e^{iuX(t)} \rangle$. A consequence of the independence of increments is that the characteristic functions are of the form $\phi_{X(t)}(u) = e^{t\psi(u)}$, where the function $\psi(u)$ is called the Lévy exponent. In order for Lévy processes to be well defined in continuous time the variables $X(t)$ need to to be infinitely divisible, and a consequence of this condition is that the Lévy exponent is on the form

$$\psi(u) = \mu u i - \frac{\sigma^2 u^2}{2} + \int\limits_{|x|\geq 1} (e^{iux} - 1) d\nu(x) + \int\limits_{|x|<1} (e^{iux} - 1 - iux) d\nu(x),$$

where ν is a measure on $\mathbb{R} \setminus \{0\}$ satisfying the condition

$$\int \min\{1, x^2\} d\nu(x) < \infty. \tag{A3}$$

The measure ν is called the Lévy measure, and if it is zero, then $X(t)$ is a Brownian motion with scale parameter σ and drift μt. If we assume that $\langle |X(1)| \rangle < \infty$, then

$$\int\limits_{|x|<1} x d\nu(x) < \infty$$

so that one can write

$$\psi(u) = (\mu - \mu') u i - \frac{\sigma^2 u^2}{2} + \int\limits_{\mathbb{R}\setminus\{0\}} (e^{iux} - 1) d\nu(x),$$

where $\mu' = \int_{|x|<1} x d\nu(x)$. Moreover, if the measure ν is finite, then one can define a probability measure $dP_J(x) = \lambda^{-1} d\nu(x)$, where

$$\lambda = \int\limits_{\mathbb{R}\setminus\{0\}} d\nu(x).$$

This gives

$$\psi(u) = (\mu - \mu') u i - \frac{\sigma^2 u^2}{2} + \lambda \int (e^{iux} - 1) dP_J(x),$$

and the corresponding process is called a jump-diffusion. If we have no Brownian component and no drift, i.e., if

$$\psi(u) = \lambda \int (e^{iux} - 1) dP_J(x),$$

then the $X(t)$ is called a Poisson jump process. The number λ is the jump rate, and $P_J(x)$ is the probability density function for the jump-size.

The increments of a Poisson jump process is a better model for the volcano forcing signal than the increments of a Lévy flight (for which λ is infinite). The reason for this is that the existence of a finite rate λ is equivalent to the assumption that there are only a finite number of volcanic eruptions in any finite time interval. The probability density $P_J(x)$ for the jumps may well be heavy-tailed, and this does not affect the singularity spectrum as long as the rate λ is finite. In fact, any such process has the following monofractal singularity spectrum (Jaffard, 1999):

$$f(\gamma) = \begin{cases} 0 & \gamma = 0 \\ -\infty & \text{else} \end{cases}$$

The results summarized in this subsection leads us to the following conclusion.

> *If the volcanic forcing signal is modeled as the increments of a Lévy process $X(t)$, then since there are only a finite number of volcanic eruptions in any finite time interval, the process $X(t)$ has a monofractal singularity spectrum.*

The discussion of wether a Lévy flight should be considered as multifractal or monofractal is hence irrelevant for the purpose of describing volcanic forcing.

Remark. Since most estimators of multifractality are based on structure functions (or similar constructions based on qth moments) it is our opinion that the definition of Mandelbrot *et al.* is the most useful. However we have to respect that other authors may use the term in different ways, and this should not be problematic as long as it is clear what is meant. In this discussion, however, we are not sure what Lovejoy and Varotsos mean by a multifractal stochastic process, and we therefore ask them to be precise on this point.