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Supplement to reply to Hébert and Lovejoy

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Abstract. I have chosen to respond to the comments by Hébert and Lovejoy by adding five appendices in the revised manuscript, which includes three more figures and one table. Fig. 1 below is Fig. 4 in the main manuscript. It is given here for easy comparison with Fig. 4.

Appendix A: Response to step forcing for one-box model

The linearised one-box model has the form

$$C_1 \frac{dT_1}{dt} = -\frac{T_1}{S_{eq}} + F. \tag{A1}$$

Here T_1 is the perturbation of the mixed-layer temperature from an imagined equilibrium and F is the forcing relative to that equilibrium. C_1 is the heat capacity per square meter of the mixed layer, and the term T_1/S_{eq} is the linearised expression for the intensity of the outgoing long-wave radiation (OLR). It is determined by the (linearised) Stefan-Boltzmann (SB) law and the effective emissivity of the atmosphere, which also contains the effects of fast feedbacks. The nonlinear version and the linearisation procduure is described in Appendix **??**. If a new equilibrium is attained with the forcing F we have

$$S_{eq} = \frac{T_1}{F},$$

which makes it natural to name S_{eq} the equilibrium climate sensitivity. It is determined from the SB constant and the effective atmospheric emissivity, i.e., it is totally determined by the atmosphere. The response function (Greensfunction: the response to $F = \delta(t)$) for the one-box model is

$$G(t) = \frac{1}{C_1} e^{-t/\tau_1} H(t), \text{ where } \tau_1 = C_1 S_{eq},$$

and $H(t)$ is the Heaviside unit step function. The response to a step-function forcing $F(t) = H(t)$ is,

$$T_1(t) = \int_{-\infty}^{t} G(t-t')\, dt' = S_{eq}(1 - e^{-t/\tau_1}). \tag{A2}$$

Appendix B: Response to step forcing for two-box model

The recent work by **?** shows that a two-exponentials response can be fitted very well to a number of 150 yr AOGCM runs with step-function forcing. This raises the question whether the power-law LRM-response representation is really only an inaccurate expression of a a response with two exponential time scales, or vice versa. There is also an issue of whether the AOGCMs really capture the true scaling properties of the observed response. The two-box model couples the mixed layer to the deep ocean temperature T_2 through a simple heat conduction term

$$\begin{aligned} C_1\frac{dT_1}{dt} &= -\frac{1}{S_{eq}}T_1-\kappa(T_1-T_2)+F \qquad \text{(B1)}\\ C_2\frac{dT_2}{dt} &= \kappa(T_1-T_2). \end{aligned}$$

where C_2 is the heat capacity of the deep ocean and κ is a heat conductivity. In the limit $C_2 \gg C_1$, the Greens-function for $T_1(t)$ correct to lowest order in the small parameter C_1/C_2, is very simple and transparent;

$$G(t)=\left(\frac{S_{tr}}{\tau_{tr}}e^{-t/\tau_{tr}}+\frac{S_{eq}-S_{tr}}{\tau_{eq}}e^{-t/\tau_{eq}}\right)H(t), \qquad \text{(B2)}$$

The reponse to a step-function forcing; $F=H(t)$ then becomes

$$T_1(t)=S_{tr}(1-e^{-t/\tau_{tr}})+(S_{eq}-S_{tr})(1-e^{-t/\tau_{eq}}), \qquad \text{(B3)}$$

where we have introduced some new parameters,

$$S_{tr}=\frac{S_{eq}}{1+\kappa S_{eq}},\ \tau_{tr}=C_1S_{tr},\ \tau_{eq}=\frac{C_2S_{eq}}{1-S_{tr}/S_{eq}}. \qquad \text{(B4)}$$

These parameters replace the heat capacities $C_{1,2}$ and the heat coupling constant κ, whose physical meaning is easy to grasp, but hard to measure directly. The meaning of the new parameters is apparent if we consider the response to a step-function forcing. Since $C_1/C_2 \ll 1$ we have $\tau_{tr} \ll \tau_{eq}$, and for $t \ll \tau_{eq}$ the response is completely dominated by the first term in equation (**??**), and hence relaxes exponentially with the transient time constant τ_{tr} to the new quasi-equilibrium S_{tr}, which is denoted the transient climate sensitivity. However, when t approaches τ_{eq} the second term comes into play, and there is a new delayed response with time constant τ_{eq} giving relaxation to the full radiative equilibrium S_{eq}.

From comparing the terms $-T_1/S_{eq}$ and $-\kappa(T_1-T_2)$ in Eq. (**??**) we observe that κS_{eq} measures the ratio between the heat flux into the deep ocean and the OLR at the early stage of the response, i.e., when T_2 is still close to zero. From Eq. (**??**) we have that the part of the sensitivity caused by the slow response from interaction with the deep ocean is

$$S_{eq}-S_{tr}=(\kappa S_{eq})\,S_{tr}.$$

Hence, it appears that κS_{eq} is an important parameter. If $\kappa S_{eq} \ll 1$ the inclusion of the deep ocean has little effect on the relaxation to equilibrium. If $\kappa S_{eq} \simeq 1$ or larger the slow response leads to a significant rise of the temperature after the transient equilibrium has been attained. The fast and the slow time constants are always well separated if $C_1 \ll C_2$ since

$$\frac{\tau_{tr}}{\tau_{eq}} = \frac{C_1}{C_2}\frac{\kappa S_{eq}}{(1+kS_{eq})^2} \leq \frac{C_1}{4C_2}.$$

Appendix C: Response to step forcing in LRM model and GCMs

The LRM-scaling response function $G_T(t) = \alpha_T t^{\beta_T/2-1}$ yields a response $T \sim t^{\beta_T/2}$ to a step in the forcing at time $t = 0$, while a linearly growing forcing yields a response $T \sim t^{\beta_T/2+1}$. Since the forcing is logarithmic in the CO_2 concentration the latter corresponds to exponentially growing concentration. Climate-model runs with linearly growing forcing are of course more realistic than step-function runs, but both have been conducted as part of the CMIP5 project. Examples are 150 yr long simulations of the GISS-E2-H model with a sudden quadrupling of the CO_2-concentration (Fig. **??**a) and a 1% per yr increase in the CO_2-concentration (Fig. **??**b). A fit of the LRM-scaling response $T \sim t^{\beta_T/2}$ to the GISS-model result in Fig. **??**a yields $\beta_T \approx 0.32$, and the solution is shown as the red curve in the figure. The solution of the form $T \sim t^{\beta_T/2+1}$ is shown as the red curve in Fig. **??**b. The fit to the tail of the step-function response looks good up to the 150 yr duration of the simulation, but the divergence of the solution as $t \to \infty$ indicates that the power-law tail with $\beta_T > 0$ is unrealistic for sufficiently large times. There exist few AOGCM simulations that investigate the response to such idealised forcing on millennium time scale. In **?** some figures with results of such runs are given. Fig. **??**c is an adaptation of Fig. 3 in **?**, which shows a 2000 yr long run of the GISS ModelE-R, and Fig. **??**d shows a plot of the function $ct^{\beta_T/2+1}$ with $\beta = 0.32$. It demonstrates that at least this particular AOGCM exhibits the power-law tail in the temperature response on time scales up to two millennia.

Note that the $\beta_T \approx 0.32$ obtained for the LRM-model on long time scales is smaller than the $\beta_T \approx 0.75$ estimated from the spectra of the residual of the instrumental data after the response to the deterministic forcing has been subtracted (**?**). If we produce such residuals by subtracting the red curves from the GISS-model curves in Fig. **??**a,b the result looks like a fractional Gaussian noise (fGn) with spectral exponent $\beta \approx 0.65$. As mentioned in Sect. 2.1 an fGn $x_\beta(t)$ characterised by the spectral exponent β is produced by the convolution integral Eq. (2) in the main manuscript if the response kernel is $G(t) \sim t^{\beta/2-1}$ and the forcing function $F(t)$ is a white Gaussian noise $x_0(t)$ (white noise is an fGn with $\beta = 0$). In other words we have,

$$x_\beta(t) = \int_{-\infty}^{\infty} t'^{\beta/2-1} H(t-t')\, x_0(t')\, dt', \tag{C1}$$

where $H(t)$ is the unit step function. By using the convolution theorem for the Fourier transform it is easily shown (**?**) that if $F(t)$ is an fGn with spectral exponent β_F, and the response function has exponent β_T, then the convolution will produce an fGn with $\beta = \beta_T + \beta_F$;

$$x_\beta(t) = \int\limits_{-\infty}^{\infty} t'^{\beta_T/2-1} H(t-t')\, x_{\beta_F}(t')\, dt'. \tag{C2}$$

In **?** it was suggested that the discrepancy between the spectral exponent β of residuals in observed and simulated GMST records could be explained by assuming some long-range memory ($\beta_F > 0$) in the stochastic forcing. It was pointed out there that this LRM could even be present in the CO_2-forcing, since some recent studies indicate strong spatiotemporal heterogeneity in the atmospheric CO_2 concentration which might give rise to a fluctuating global component of the global CO_2-forcing with long-memory properties.

Appendix D: Two-box vs. LRM fitting to GCM results

? have fitted the two-box model to 16 runs of 150 yr length to step-function forcing. There are four fitting parameters, and the fits are generally good. There is, however, a wide scatter in the fitting parameters between the different models, which may be an indication of overfitting. In Fig. **??** the surface tempeature solution to the two-box model

$$T_1(t) = [S_{tr}(1-\exp(-t/\tau_{tr})) + (S_{eq}-S_{tr})(1-\exp(-t/\tau_{eq}))]\, F_{4\times \mathrm{CO_2}}, \tag{D1}$$

and to the LRM model

$$T_1(t) = ct^{\beta_T/2} F_{4\times \mathrm{CO_2}}, \tag{D2}$$

have been fitted to simulation results for the GMST of climate models with step-forcing, $F(t) = F_{4\times \mathrm{CO_2}} H(t)$. Here $F_{4\times \mathrm{CO_2}} \approx 8.61$ Wm^{-2} is the forcing associated with a quadrupling of the atmospheric CO_2 concentration. The fitting parameters obtained are given in Table 1.

Model	τ_1 (months)	τ_2 (months)	S_{tr} (Km^2/W)	S_{eq} (Km^2/W)	c	β_T
GISS-E2-H	26	663	0.29	0.46	0.14	0.32
BNU-ESM	46	729	0.46	0.69	0.21	0.33
CCSM4	49	4.1×10^{10}	0.33	3.9×10^6	0.10	0.40
CNRM_CM5	38	390	0.37	0.58	0.20	0.31
MPI-ESM-LR	34	1061	0.46	0.75	0.20	0.33

Table 1. Parameters estimated by fitting Eqs. (**??**) and (**??**) to the climate model responses to an abrupt quadrupling of atmospheric CO_2 shown in Fig. **??**. The table shows the parameters obtained by the Mathematica routine FindFit.

The LRM-model in general gives a poorer fit on the short time scales. This is not surprising, since the LRM-response $ct^{\beta_T/2}$ has an infinite derivative at $t = 0$. However, a much better approximation is obtained if we fit the LRM model only in the interval $(0, 100)$ months, but then β_T is raised to approximately 0.75. If we implement a four-parameter model with one power-law ($\beta_T \approx 0.75$) up to 100 months and another ($\beta_T \approx 0.35$) for $t > 100$ months, we obtain fits comparable to the two-exponential model. There is a wide scatter in the model parameters for the two-box model. Note particularly the huge values for τ_{eq} and S_{eq} for the CCSM4 model. The long time-scale tail is not captured by a reasonable exponential, but is well approximated by a reasonable power-law. On the other hand, the scatter in the LRM-model parameters is small. All this indicates that the two-box model may suffer from overfitting in some cases.

When projections are limited to 2200 CE there is no practical difference between using a power-law response kernel (the LRM model) and the two-exponential kernel (the two-box model). This is illustrated in Fig. **??**, where we compute the response for the exponential CO_2-concentration model with $\tau_C = 33$ yr and the two-box model parameters corresponding to the GISS-E2-H model and the CNRM_CM5 models, respectively. The parameters for the two models differ significantly, but the projections are almost identical. Morever, they are very similar to the projections in Fig. **??**a, where the temperature response is produced by the LRM-model with $\tau_C = 33$ yr and $\beta_T = 0.35$. This demonstrates that the mathematical divergence of the solution Eq. (**??**) for a step-function forcing has little impact on the projection up to 2200 CE for the forcing scenarios considered here. The advantage of the power-law kernel is that it provides a more parsimonious description (fewer fitting parameters) which provides a more precise parameter estimation.

Appendix E: Divergences, causality and initial conditions

If $G(t)$ is a power law the integral over prehistory $t \in (-\infty, 0)$ may lead to paradoxes, such as divergences of the integral. The solution to the paradox is to interpret the power-law as an approximation, for instance to a superposition of exponential response kernels. For a white-noise forcing this corresponds to an aggregation of Ornstein-Uhlenbeck (OU) processes, which are known to have the potential to produce a process that is a very good approximation to a fractional Gaussian noise (fGn) up to the time scale corresponding to the OU process with the greatest correlation time (**?**).

The scaling properties on scales of decades and longer arise from the heat transport within the oceans. This transport exhibits a maximum response time, which will provide an upper (exponential) cut-off of the power-law response function, but the characteristic time of this cut-off may be centuries or millennia. **?** state in their abstract: “Scaling up to decades is demonstrated in observations and coupled atmosphere-ocean models with complex and mixed-layer oceans. Only with the complex ocean model the simulated power laws extend up to centuries.”

If we don't treat the power-law as an approximation we have to deal with the divergences of the integral

$$\Delta T(t) = \int_{-\infty}^{t} G(t-t')\, F(t')\, dt', \tag{E1}$$

where $G(s) = s^{\beta_T/2-1}$. If we consider the unit step-function forcing $F(t) = H(t)$, and $\beta_T \neq 0$, the integral is

$$\Delta T(t) = \lim_{\epsilon \to 0^+} \int_{\epsilon}^{t} (t-t')^{\beta_T/2-1}\, dt' = \lim_{\epsilon \to 0^+} \int_{\epsilon}^{t} s^{\beta_T/2-1}\, ds = \lim_{\epsilon \to 0^+} \frac{2}{\beta_T}(t^{\beta_T/2} - \epsilon^{\beta_T/2}). \tag{E2}$$

Clearly $\Delta T(t)$ diverges as $t \to \infty$ if $\beta_T > 0$, but it also diverges if $\beta_T < 0$ (as $\epsilon \to 0^+$). For $\beta_T = 0$ there is a logarithmic divergence in both limits.

For physically meaningful results the $\beta_T > 0$ case requires some sort of cut-off (e.g., an exponential tail) for sufficiently large t, and the $\beta_T < 0$ case requires an elimination of the strong singularity of $G(s)$ at $s = 0$. A shown in Appendix **??**, AOGCMs in the CMIP5 ensemble with step function forcing indicate a power-law response for large s at least up to 150 yr (and the GISS-E2-R model up to 2000 yr) with $\beta_T \approx 0.35$, so $\beta_T > 0$ is the case of interest for the global temperature response. The AOGCMs are also well approximated by an exponential response in the limit $s \to 0$ (for s up to a few years), so an exponential truncation in this high-frequency limit is also appropriate.

The truncation of the power-law kernels is a physical, and not a technical mathematical issue. It is an approximation to a hierarchy of exponential responses. With this interpretation the divergences evaporate. Below is a more detailed outline of this philosophy in an energy-balance context. Let us take as a starting point the simple zero-dimensional EBM before linearisation of the Stefan-Boltzmann law;

$$C\frac{dT}{dt} = -\epsilon\sigma_S T^4 + I(t), \tag{E3}$$

where T is surface temperature in Kelvin, C is an effective heat capacity per area of the Earth's surface, σ_S is the Stefan-Boltzmann constant, ϵ is an effective emissivity of the atmosphere, and $I(t)$ is the incoming radiative flux density at the top of the atmosphere. Let $I_0 = I(0)$ be the initial incoming flux, $F(t) = I(t) - I_0$ is the radiative forcing, $T_{eq} = (I_0/\epsilon\sigma_S)^{1/4}$ is the equilibrium temperature at $t = 0$, $\Delta T(t) = T(t) - T_{eq}$ is the temperature anomaly measured relative to the initial equilibrium temperature, and $\Delta T_0 = \Delta T(0)$ is this anomaly at $t = 0$. Note that F here is the perturbation of the radiative flux with respect to the initial flux I_0 and not with respect to the flux $\epsilon\sigma_S T_0^4$ that would be in equilibrium with the initial temperature T_0. The linearised EBM for the temperature change relative to the temperature T_0 (the one-box model) is

$$\frac{d\Delta T}{dt} = -\nu\Delta T + \mathcal{F}(t), \quad \Delta T(0) = \Delta T_0, \tag{E4}$$

where $\nu = 4\epsilon\sigma_S T_0^3/C$, $\mathcal{F}(t) = F(t)/C$. By definition $\mathcal{F}(0) = [I(0) - I_0]/C = 0$. This is Eq. (**??**) and Eq. (**??**) with slightly different notation. The solution the initial value problem (i.v.p.) Eq. (**??**), with the initial condition $\Delta T(0) = \Delta T_0$, takes the form

$$\Delta T_{\text{i.v.p.}} = \int_0^t G(t-t')\mathcal{F}(t')\,dt' + \Delta T_0 e^{-\nu t}, \tag{E5}$$

where $G(s) = \exp(-\nu s)$. The generalisation to a linear, causal response model, where $G(s)$ is not necessarily exponential, involves extending the integration domain in Eq. (**??**) to the interval $(-\infty, t)$;

$$\Delta T_{\text{r.m.}}(t) = \int_{-\infty}^t G(t-t')\mathcal{F}(t')\,dt'. \tag{E6}$$

From the initial condition $\Delta T(0)_{\text{r.m.}} = \Delta T_0$ Eq. (**??**) yields,

$$\Delta T_0 = \int_{-\infty}^0 G(-t')\mathcal{F}(t')\,dt'. \tag{E7}$$

For exponential response $G(s) = \exp(-\nu s)$ it is easy to verify that $\Delta T_{\text{i.v.p.}}(t) = \Delta T_{\text{r.m.}}(t)$, and Eq. (**??**) yields the following relation between the initial temperature anomaly and the forcing $\mathcal{F}(t)$ for $t \in (t, 0)$;

$$\Delta T_0 = \int_{-\infty}^0 e^{\nu t'}\mathcal{F}(t')\,dt'. \tag{E8}$$

For the exponential response there is no "divergence issue" in Eq. (**??**). Neither is there such an issue for the two-exponential solution to the two-box model (**?**). An "N-box model" exhibits a response function for the temperature in each box which is a superposition of exponentials; $G(s) = \sum_{i=1}^N a_i \exp(-\nu_i s)$. For the surface (mixed layer) box the temperature anomaly takes the form

$$\Delta_{\text{r.m.}}(t) = \sum_{i=1}^N a_i e^{-\nu_i t} \int_{-\infty}^t e^{\nu_i t'}\mathcal{F}(t')\,dt'. \tag{E9}$$

On the other hand, the N-box initial value problem has solution of the form

$$\Delta T_{\text{i.v.p.}}(t) = \sum_{i=1}^N a_i e^{-\nu_i t} \int_0^t e^{\nu_i t'}\mathcal{F}(t')\,dt' + \sum_{i=1}^N b_i e^{-\nu_i t}, \tag{E10}$$

where the coefficients b_i are linearly related to the initial temperatures of each box; $b_i = \sum_{j=1}^N M_{ij} T_{0j}$. The condition $\tilde{T}_{\text{i.v.p.}}(t) = \tilde{T}_{\text{r.m.}}(t)$ now yields the relations between the initial temperatures and the prehistory of the forcing;

$$\sum_{j=1}^N M_{ij}\Delta T_{0j} = a_i \int_{-\infty}^0 e^{\nu_i t'}\mathcal{F}(t')\,dt' \;\text{ for }\; i = 1, \ldots, N. \tag{E11}$$

With a white-noise forcing $\mathcal{F}(t)$ the Eq. (**??**) is the Itô stochastic differential equation (in physics often called the Langevin equation). The solution is the Ornstein-Uhlenbeck (OU) stochastic process, which in discrete time corresponds to the first-order autoregressive (AR(1)) process. The power spectral density of this process is essentially a Lorentzian, which means that the high-frequency ($f \gg \nu$) part of the spectrum has the form $\sim f^{-2}$, and the low-frequency part $\sim f^{0}$. This means that if the climate response were well described by a one-box EBM we could use a power-law response model with $\beta_T \approx 2$ on time scales much shorter than the correlation time $\tau_c = \nu^{-1}$. On these time scales the stochastic process exhibits the characteristics of a Brownian motion (Wiener process), which is a self-similar process with spectral index $\beta = 2$. This process is non-stationary, and hence suffers from the divergences that we are worried about. But even though the Brownian motion diverges, the OU-process does not, because of the flattening of the spectrum for $f \ll \nu$.

Both observation data and AOGCMs indicate that the one-box EBM is inadequate, but the considerations above are equally valid for an N-box model, for which the white-noise forcing gives rise to an aggregation of OU-processes with different ν_i. Such an aggregation is known to be able to produce a process with approximate power-law spectrum with $0 < \beta < 2$ on time scales $\tau < \nu_{\text{min}}^{-1}$ (**?**).

? specifically argue that volcanic forcing may have a scaling exponent $\beta_F \approx 0.4$, and hence the convergence criterion $\beta = \beta_T + \beta_f < 1$ then requires $\beta_T < 0.6$. One remark to this is that the above discussion shows that the $\beta < 1$ criterion is not necessary on time scales shorter than $\tau < \nu_{\text{min}}^{-1}$. However, observation indicates that $\beta < 1$, so this does not invalidate their argument. More important is that in recent papers the response to volcanic forcing has been subtracted from both instrumental and multiproxy reconstruction data **?** and from millennium-long AOGCM simulations (**?**), and the residuals have been analysed for β without finding a detectable influence of the volcanic forcing on β. The same is seen by comparing control runs of the AOGCMs with those driven by volcanic forcing (**?**).

Appendix F: Non-stationarity of the CO_2 response

In Sect. 2.2 we found (by comparing Figs. **??**b and **??**c that the LRM CO_2 response with $\beta_C = 1.6$ gives approximately the same evolution of CO_2 concentration up to 2200 CE as a response where 50% of the emitted CO_2 is absorbed by the surface almost immediately and the remainder decays exponentially with a time constant $\tau_C = 300$ yr. This is analogous to the situation with the temperature response, where where an LRM response gives very similar results to a two-exponential response with appropriate fitting of model parameters (see Appendix **??**. The most important difference is that the β_C-parameter is larger than unity. A step-function emission rate $R(t) = H(t)$ will give rise to a CO_2 concentration that grows like $(2\alpha_T/\beta_C)t^{\beta_C/2}$. This non-stationarity (divergence) of the response as $t \to \infty$ is reasonable, since the surface will not be able to absorb a sufficient fraction of the constantly emitted CO_2 to establish a new equilibrium. The exponential response kernel Eq. (**??**), on the other hand yields the response $r[1-\exp(-t/\tau_C)]$ to the step forcing. This implies establishment of a new equilibrium CO_2-concentration after $t \gg \tau_C$. This has little consequence as long as we consider projection only up to 2200 CE (and $\tau_C \approx 300$ yr). On millennium time scale we have the positive ice-age feedback, by which warming may lead to net release of CO_2 to the atmosphere, and hence lead to continuing growth of CO_2 concentration. It is assumed to be important in the triggering of glacial-interglacial transitions, although it is not very well understood. On time scales of hundreds of kyr we have the negative Carbon weathering-cycle feedback that will eventually lead to a Carbon cycle equilibrium. The most interesting feature of this feedback in the present context is that it suggests that the anthropogenic global warming event may last for such a long time in absence of effective Carbon sequestration measures (**?**).

A more problematic non-stationarity of the Carbon-cycle response arises from stochastic forcing. In this case the power-law response function will give rise to a fractional Brownian motion (fBm) with power-spectral index $\beta_C \approx 1.6$. This is a non-stationary stochastic process in the sense that the variance increases with time as $t^{\beta_C - 1}$, which is not physically reasonable for sufficiently large t. Here we may be saved by an exponential cut-off of the power-law tail, but this requires some sort of negative Carbon-cycle feedback. It is difficult to assess the magnitude of the natural stochastic component of the CO_2 emission rate. If it small the weathering-cycle feedback may be sufficient.

References

Fraedrich, K., and Blender, R., Scaling of Atmosphere and Ocean Temperature Correlations in Observations and Climate Models:. *Phys. Rev. Lett.*, *90*, 108501-4, doi:10.1103/PhysRevLett.90.108501, 2003.

Geoffroy, O., Saint-Martin, D., Olivié, D. J. L., Voldoire, A., Bellon, G., and Tytcá, S.: Transient Climate Response in a Two-Layer Energy-Balance Model. Part I: Analytical Solution and Parameter Calibration Using CMIP5 AOGCM Experiments, *J. Climate*, *6*, 1841-1857, doi:10.1175/JCLI-D-12-00195.1, 2013.

Granger, C. W. J.: Long Memory Relationships and the aggregation of dynamical models , *J. Econometrics*, *14*, 227-238, 1980.

Hansen, J., Sato, M., Kharecha, P., von Schuckmann, K.: Earth's energy imbalance and implications, Atmos. Chem. Phys., 11, doi:10.5194/acp-11-13421-2011, 2011.

Lovejoy, C., Schehrtzer, D., and Varon, D.: Do GCMs predict climate... or macroweather? , *Earth Syst. Dynam.*, *4*, 439-454, doi: 10.5194/esd-4-439-2013, 2013.

Rypdal, M., and Rypdal, K.: Long-memory effects in linear-response models of Earth's temperature and implications for future global warming, J. Climate, 27, 5240-5258, doi:10.1175/JCLI-D-13-00296.1, 2014.

Rypdal, K., Rypdal, M., and H.-B. Fredriksen: Spatiotemporal Long-Range Persistence in Earth's Temperature Field: Analysis of Stochastic-Diffusive Energy Balance Models, J. Climate, 28, 8379- 8395, 2015, DOI: 10.1175/JCLI-D-15-0183.1.

Østvand, L., Nilsen, T., Rypdal, K., Divine, D., and Rypdal, M.: Long-range memory in internal and forced dynamics of millennium-long climate model simulations, Earth. Syst. Dynam., 5, 295-308, doi:10.5194/esd-5-295-2014, 2014.

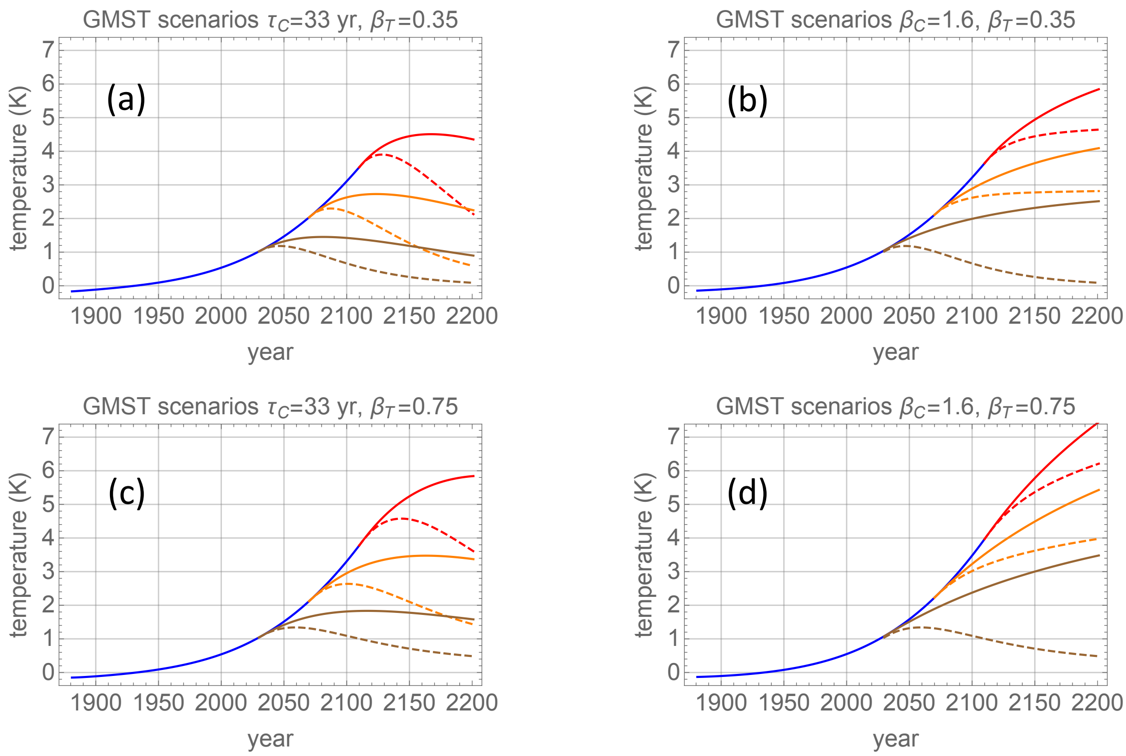

Figure 1. This is Fig. 5 in the main manuscript. (a): The evolution of the GMST for the CO_2 concentration scenarios shown in Fig. 4a and Fig. 4c in the main manuscript. (a): $\tau_C = 33$ yr and $\beta_T = 0.35$. (b): $\beta_C = 1.6$ and $\beta_T = 0.35$. (c): $\tau_C = 33$ yr and $\beta_T = 0.75$. (d): $\beta_C = 1.6$ and $\beta_T = 0.75$.

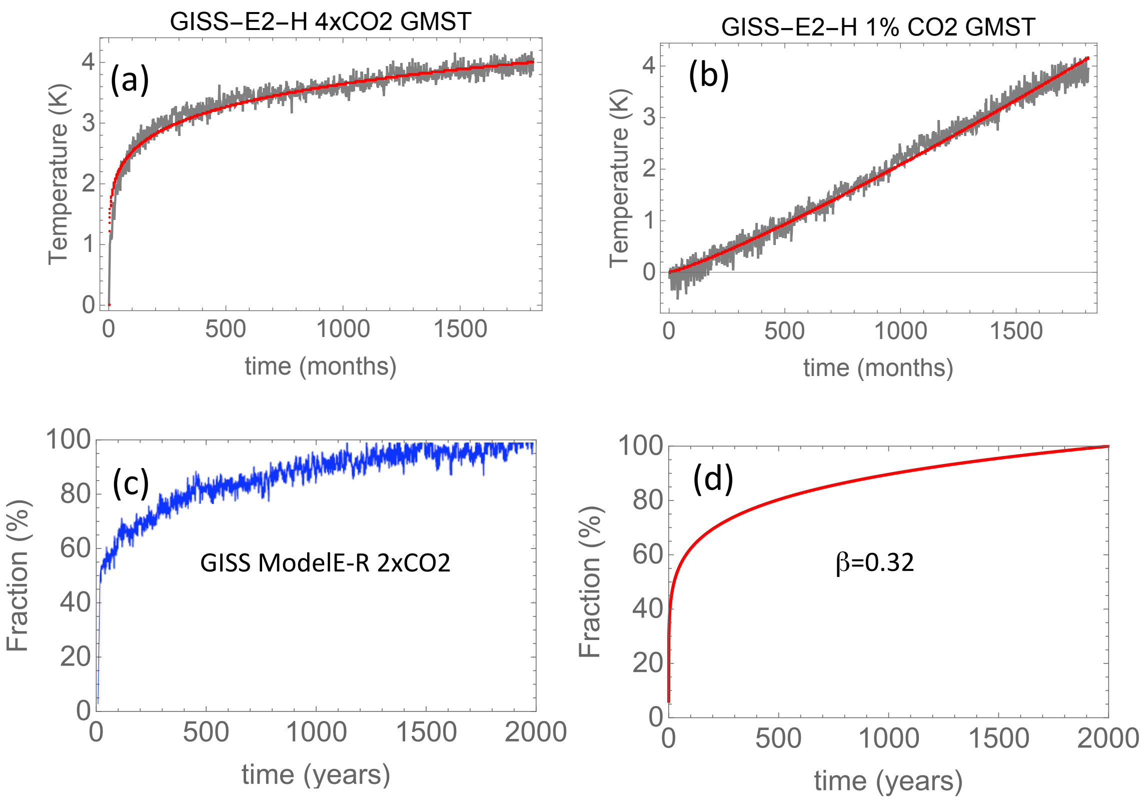

Figure 2. (a): LRM response model fit $c_1 t^{\beta_T/2}$ (red) to the GISS-E2-H model response to an abrupt quadrupling of atmospheric CO_2 (grey). The fit yields $\beta_T = 0.32$. (b): The LRM-reponse model solution $c_2 t^{\beta_T/2+1}$ with $\beta_T = 0.32$ (red) and the GISS-E2-H model response to a 1 % per yr increase in atmospheric CO_2-concentration. (c): The 2000 yr response to a doubling of CO_2 in GISS ModelE-R as taken from Figure 3 in **?**. (d) Response to the same forcing in the LRM model with $\beta_T = 0.32$.

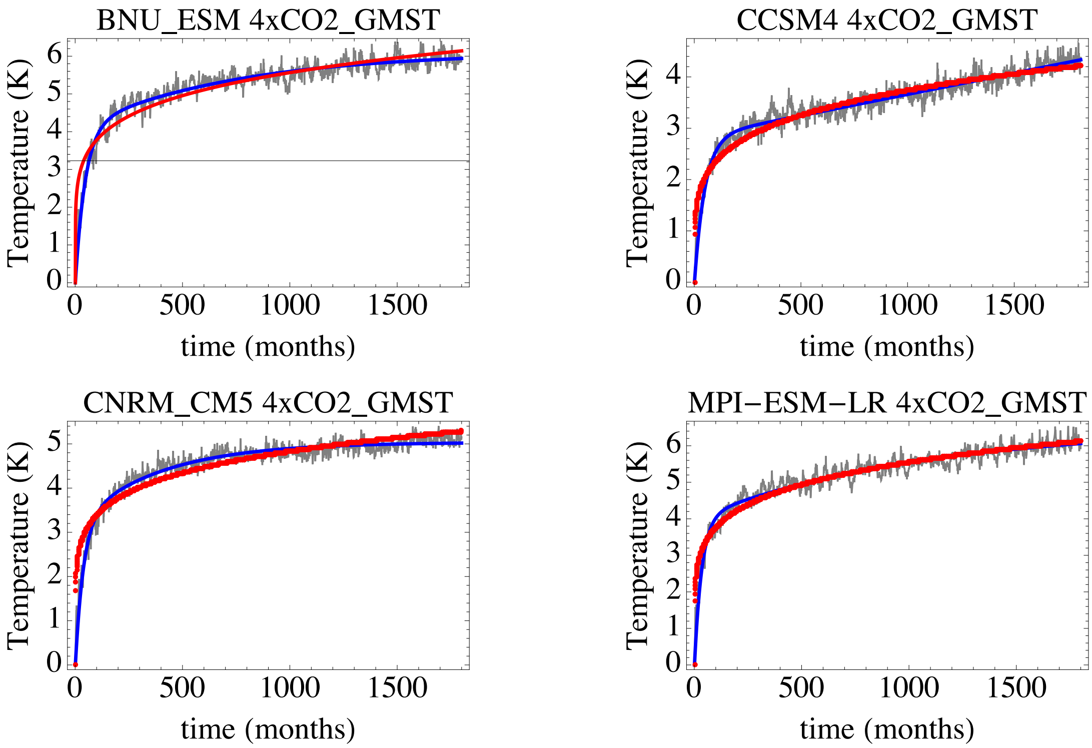

Figure 3. Blue curves: Fit of the two-exponential response to the climate model responses to an abrupt quadrupling of atmospheric CO_2 concentration. Red curves: Fit of the LRM-scaling response. The expressions fitted are found in the caption of Table 1 and the coefficients estimated are shown in this table.

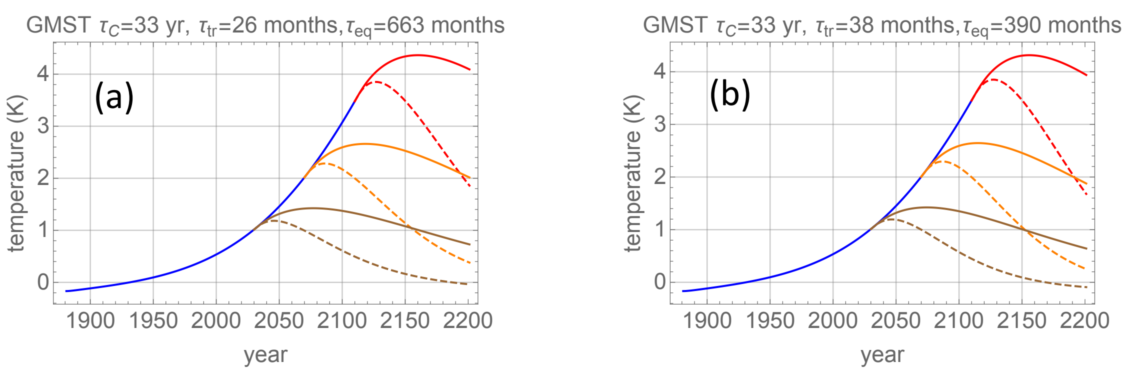

Figure 4. (a): The evolution of the GMST according to the two-box solution Eq. (**??**) for the CO_2 concentration scenarios shown in Fig. 4a and Fig. 4c in the main manuscript. (a): $\tau_C = 33$ yr and and the two-box parameters for the GISS-E2-H given in Table 1. (b): $\tau_C = 33$ yr and and the two-box parameters for the CNRM_CM5 model given in Table 1.