Earth Syst. Dynam. Discuss., 6, C1028–C1037, 2016
www.earth-syst-dynam-discuss.net/6/C1028/2016/

Earth System Dynamics Discussions

Interactive comment on "Global warming projections derived from an observation-based minimal model" *by* K. Rypdal

dr. Rypdal

kristoffer.rypdal@uit.no

Received and published: 7 January 2016

The comment by Hébert and Lovejoy (H&L) raises a number of issues which would be interesting to discuss in depth, although some have little relevance to the present discussion paper. The main comment, which *is* relevant, is the following: Will the divergence of the convolution integral over a power-law kernel with $\beta > 0$ as $t \to \infty$ have an effect on the GMST-projection up to 2200 CE? The short answer is that it does not. In the attached supplement I demonstrate this. It is written as a group of appendices which will be added in the revised manuscript.

The comment of H&L is structured in sections. Below I will respond to each section in

the given order. For more detail I refer to the Supplement attached to this reply.

1. Introduction

It is my impression that H&L base their physical interpretation of the power-law response kernel on a turbulent cascade paradigm for atmospheric dynamics (see e.g., Lovejoy and Schertzer, 2013, Chapter 1). The inertia effects (long-range memory) I am discussing in the paper do not arise from the atmosphere but from heat transport in the ocean, and possibly involves other high inertia components of the climate system, such as the cryosphere. This fact is well documented in AOGCMs, since the long-memory scaling only takes place if full ocean circulation is included. GCMs including only the mixed ocean layer does not exhibit this scaling (*Fraedrich and Blender*, 2003).

My interpretation of the approximate scale-invariance of the temperature response is much simpler and based on energy-balance considerations. The climate system consists of a number of interacting subsystems with different response times. The simplest model is the so-called two-box model, where the zero-dimensional energy balance model is supplemented with an equation describing the heat exchange between the mixed ocean layer and the deep ocean. This gives rise to a response kernel consisting of a sum of two exponentials with one short and one long e-folding time. This kernel is already well approximated by a power law up to time scales comparable to the long time constant. In the supplement I show examples from CMIP5 experiments with step-function forcing, where the response is approximated by a power-law as well as two exponentials. Using the model parameters estimated from these experiments on the forcing scenarios used in my manuscript, I demonstrate that the two kinds of response kernels give almost identical results. Since the two-exponential kernel does not suffer from the divergence problem, this shows that this divergence does not influence the projections on the time scales considered.

2. A Scaling Climate Response Function

In this section the paragraph below does not make sense to me:

> "Notice that the initial conditions are important: the same linear operator with different initial conditions will lead to a different Green's function. Due to the scaling symmetry respected by the dynamics, and boundary conditions, over a wide range of scales, we take the basic Green's function as a power law."

The integral goes from $-\infty$ to t, so what does the "initial condition" mean? Let us define a time origin ($t = 0$) at a specific date, and define T and its derivatives at this date as the initial conditions. The mathematical fact is then that if $\mathcal{L}$ is an ordinary n'th order differential operator the evolution for $t > 0$ is uniquely determined by the $T(0)$ and the $n-1$ first derivatives at $t = 0$. On the other hand, I will show in Section 3 below that these initial conditions in turn depend on the entire prehistory of the forcing in the time interval $t \in (-\infty, 0)$. The Green's function is determined by the dynamics, is not by the initial conditions. On the contrary, the initial conditions are determined by the Green's function and the past forcing.

However, if $\mathcal{L}$ is a fractional derivative (i.e., if $G(t)$ is a power law), then the integral over prehistory $t \in (-\infty, 0)$ may lead to paradoxes, such as divergences of the integral. The solution to the paradox is to interpret the power-law as an approximation, for instance to a superposition of exponential kernels. For a white-noise forcing this corresponds to an aggregation of Ornstein-Uhlenbeck (OU) processes, which are known to have the potential to produce a process that is a very good approximation to a fractional Gaussian noise (fGn) up to the time scale corresponding to the OU process with the

greatest correlation time. More on this in Section 3.

As discussed in the previous section, the scaling properties on scales of decades and longer do not arise from "the scaling symmetry respected by the (atmospheric) dynamics, and boundary conditions", but from the heat transport within the oceans. This transport exhibits a maximum response time, which will provide an upper (exponential) cut-off of the power-law response function, but the characteristic time of this cut-off may be centuries or millennia. Let me cite from the abstract of *Fraedrich and Blender* (2003): "Scaling up to decades is demonstrated in observations and coupled atmosphere-ocean models with complex and mixed-layer oceans. Only with the complex ocean model the simulated power laws extend up to centuries."

3. Low-frequency divergence

In the present paper deterministic (anthropogenic) forcing is the only forcing considered. As discussed in *Rypdal and Rypdal* (2014) the internal variability can be taken into account as a result of stochastic forcing whose variance can be estimated from the observation data. This allows to compute error bars due to internal variability (stochastic errors) on predictions based on the minimal model. I haven't included such error estimates in the paper because I consider model uncertainty to be more important.

The H&L comment discusses the conditions for divergence of the integral

$$T(t) = \int_{-\infty}^{t} G(t-t')\, F(t')\, dt', \tag{1}$$

where $G(s) = s^{\beta_T/2-1}$. If we consider the unit step-function forcing $F(t) = H(t)$ the integral is

$$T(t) = \lim_{\epsilon\to 0^+}\int_\epsilon^t (t-t')^{\beta_T/2-1}\,dt' = \lim_{\epsilon\to 0^+}\int_\epsilon^t s^{\beta_T/2-1}\,ds = \lim_{\epsilon\to 0^+}\frac{2}{\beta_T}(t^{\beta_T/2}-\epsilon^{\beta_T/2}), \quad (2)$$

for $\beta_T \neq 0$. Clearly $T(t)$ diverges as $t\to\infty$ if $\beta_T > 0$, but it also diverges if $\beta_T < 0$ (as $\epsilon \to 0^+$). For $\beta_T = 0$ there is a logarithmic divergence in both limits.

For physically meaningful results the $\beta_T > 0$ case requires some sort of cut-off (e.g., an exponential tail) for sufficiently large t, and the $\beta_T < 0$ case requires an elimination of the strong singularity of $G(s)$ at $s = 0$. AOGCMs in the CMIP5 ensemble with step function forcing indicate a power-law response for large s at least up to 150 yr (and the GISS-E2-R model up to 2000 yr) with $\beta_T \approx 0.35$, so $\beta_T > 0$ is the case of interest for the global temperature response. As discussed in Sect. 1 above (and in the Supplement) the inclusion of such a cut-off consistent with the AOGCM results will have very small effects on the predictions up to 2200 CE. The AOGCMs show an exponential response for $s \to 0$ (for s up to a few years), so an exponential truncation in this high-frequency limit is also appropriate.

I agree with H&L that the truncation of the power-law kernels is a "physical, and not a technical mathematical issue." But their comment is nothing but a mathematical discussion of convergence, detached from the physical interpretation that I have given to the power-law response as an approximation to a hierarchy of exponential responses. With this interpretation the divergences evaporate. Below I outline the philosophy in some detail in an energy-balance context. Let us take as a starting point the simple zero-dimensional EBM before linearisation of the Stefan-Boltzmann law;

$$C\frac{dT}{dt} = -\epsilon\sigma_S T^4 + I(t), \quad (3)$$

where T is surface temperature in Kelvin, C is an effective heat capacity per area of the Earth's surface, σ_S is the Stefan-Boltzmann constant, ϵ is an effective emissivity of the atmosphere, and $I(t)$ is the incoming radiative flux density at the top of the atmosphere. Let $I_0 = I(0)$ be the initial incoming flux, $F(t) = I(t) - I_0$ is the radiative forcing, $T_{eq} = (I_0/\epsilon\sigma_S)^{1/4}$ is the equilibrium temperature at $t = 0$, $\tilde{T}(t) = T(t) - T_{eq}$ is the temperature anomaly measured relative to the initial equilibrium temperature, and $\tilde{T}_0 = \tilde{T}(0)$ is this anomaly at $t = 0$. Note that F here is the perturbation of the radiative flux with respect to the initial flux I_0 and not with respect to the flux $\epsilon\sigma_S T_0^4$ that would be in equilibrium with the initial temperature T_0. The linearised EBM for the temperature change relative to the temperature T_0 (the one-box model) is

$$\frac{d\tilde{T}}{dt} = -\nu\tilde{T} + \mathcal{F}(t), \quad \tilde{T}(0) = \tilde{T}_0. \quad (4)$$

where $\nu = 4\epsilon\sigma_S T_0^3/C$, $\mathcal{F}(t) = F(t)/C$. By definition $\mathcal{F}(0) = [I(0) - I_0]/C = 0$. The solution the initial value problem (i.v.p.) Eq. (4), with the initial condition $\tilde{T}(0) = T_0$, takes the form

$$\tilde{T}_{i.v.p.} = \int_0^t G(t-t')\mathcal{F}(t')\,dt' + T_0 e^{-\nu t}, \quad (5)$$

where $G(s) = \exp(-\nu s)$. The generalisation to a linear, causal response model, where $G(s)$ is not necessarily exponential, involves extending the integration domain in Eq. (5) to $-\infty$;

$$\tilde{T}_{r.m.}(t) = \int_{-\infty}^t G(t-t')\mathcal{F}(t')\,dt', \quad (6)$$

which corresponds to Eq. (5) in the H&L comment. From the initial condition $\tilde{T}(0)_{r.m.} = \tilde{T}_0$ Eq. (6) yields

$$\tilde{T}_0 = \int_{-\infty}^0 G(-t')\mathcal{F}(t')\,dt'. \quad (7)$$

For exponential response $G(s) = \exp(-\nu s)$ it is easy to verify that $\tilde{T}_{i.v.p.}(t) = \tilde{T}_{r.m.}(t)$, and Eq. (7) yields the following relation between the initial temperature anomaly and the past forcing $\mathcal{F}(t)$ for $t < 0$;

$$T_0 = \int_{-\infty}^{0} e^{\nu t'} \mathcal{F}(t')\, dt'. \tag{8}$$

For the exponential response there is no "divergence issue" in Eq. (6). Neither is there such an issue for the two-exponential solution to the two-box model (*Geoffroy et al.*, 2013). An "N-box model" exhibits a response function for the temperature in each box which is a superposition of exponentials; $G(s) = \sum_{i=1}^{N} a_i \exp(-\nu_i s)$. For the surface (mixed layer) box the temperature anomaly takes the form

$$\tilde{T}_{r.m.}(t) = \sum_{i=1}^{N} a_i e^{-\nu_i t} \int_{-\infty}^{t} e^{\nu_i t'} \mathcal{F}(t')\, dt'. \tag{9}$$

On the other hand, the N-box initial value problem has solution of the form

$$\tilde{T}_{i.v.p.}(t) = \sum_{i=1}^{N} a_i e^{-\nu_i t} \int_{0}^{t} e^{\nu_i t'} \mathcal{F}(t')\, dt' + \sum_{i=1}^{N} b_i e^{-\nu_i t}, \tag{10}$$

where the coefficients b_i are linearly related to the initial temperatures of each box; $b_i = \sum_{j=1}^{N} M_{ij} T_{0j}$. The condition $\tilde{T}_{i.v.p.}(t) = \tilde{T}_{r.m.}(t)$ now yields the relations between the initial temperatures and the prehistory of the forcing;

$$\sum_{j=1}^{N} M_{ij} T_{0j} = a_i \int_{-\infty}^{0} e^{\nu_i t'} \mathcal{F}(t')\, dt' \quad for \quad i = 1, \ldots, N. \tag{11}$$

With a white-noise forcing $\mathcal{F}(t)$ the Eq. (4) is the Itô stochastic differential equation (in physics often called the Langevin equation). The solution is the Ornstein-Uhlenbeck

(OU) stochastic process, which in discrete time corresponds to the first-order autoregressive (AR(1)) process. The power spectral density (PSD) of this process is essentially a Lorentzian, which means that the high-frequency ($f \gg \nu$) part of the spectrum has the form $\sim f^{-2}$, and the low-frequency part $\sim f^{0}$. This means that if the climate response were well described by a one-box EBM we could use a power-law response model with $\beta_T \approx 2$ on time scales much shorter than the correlation time $\tau_c = \nu^{-1}$. On these time scales the stochastic process exhibits the characteristics of a Brownian motion (Wiener process), which is a self-similar process with spectral index $\beta = 2$. This process is non-stationary, and hence suffers from the divergences that worries L&H. But even though the Brownian motion diverges, the OU-process does not, because of the flattening of the spectrum for $f \ll \nu$.

L&H argue that if the forcing is not white, but characterised by a spectral exponent β_f, then the criterion for convergence of the response requires $\beta = \beta_T + \beta_f < 1$. But this criterion is only necessary for the long time scales where the temperature process has a stationary character (in the one-box example on time scales greater than τ_c). If the time scales we study are shorter, we may accept $\beta > 1$.

Both observation data and AOGCMs indicate that the one-box EBM is inadequate, but the considerations above are equally valid for an N-box model, for which the white-noise forcing gives rise to an aggregation of OU-processes with different ν_i. Such an aggregation is known to be able to produce a process with approximate power-law spectrum with $0 < \beta < 2$ on time scales $\tau < \nu_{min}^{-1}$ (*Granger*, 1980).

L&H specifically argue that volcanic forcing may have a scaling exponent $\beta_f \approx 0.4$, and that the convergence criterion $\beta = \beta_T + \beta_f < 1$ then requires $\beta_T < 0.6$. One remark to this is that the above discussion shows that the $\beta < 1$ criterion is not necessary on time scales shorter than $\tau < \nu_{min}^{-1}$. However, observation indicates that $\beta < 1$, so this does not invalidate L&H's argument. More important is that the response to volcanic forcing has been subtracted from both instrumental and multiproxy reconstruction data *Rypdal and Rypdal* (2014) and from millennium-long AOGCM simulations (*Østvand et al.*, 2014), and the residuals have been analysed for β without finding a detectable influence of the volcanic forcing on β. The same is seen by comparing control runs of the AOGCMs with those driven by volcanic forcing (*Østvand et al.*, 2014).

There are a number of problems with the stochastic characterisation of volcanic forcing and how such a shot-noise like process could be incorporated in a response model. This is an interesting issue, but too far from the subject of the present paper to be discussed further here.

References

Fraedrich, K., and Blender, R., Scaling of Atmosphere and Ocean Temperature Correlations in Observations and Climate Models:. *Phys. Rev. Lett.*, *90*, 108501-4, doi:10.1103/PhysRevLett.90.108501, 2003.

Geoffroy, O., Saint-Martin, D., Olivié, D. J. L., Voldoire, A., Bellon, G., and Tytcá, S.: Transient Climate Response in a Two-Layer Energy-Balance Model. Part I: Analytical Solution and Parameter Calibration Using CMIP5 AOGCM Experiments, *J. Climate*, *6*, 1841-1857, doi:10.1175/JCLI-D-12-00195.1, 2013.

Granger, C. W. J.: Long Memory Relationships and the aggregation of dynamical models , *J. Econometrics*, *14*, 227-238, 1980.

Hansen, J., Sato, M., Kharecha, P., von Schuckmann, K.: Earth's energy imbalance and implications, Atmos. Chem. Phys., 11, doi:10.5194/acp-11-13421-2011, 2011.

Lovejoy, S., and Schertzer, D.: The Weather and Climate, Emergent laws and Multifractal Cascades, Cambridge University Press, 2013.

Rypdal, M., and Rypdal, K.: Long-memory effects in linear-response models of Earth's temperature and implications for future global warming, J. Climate, 27, 5240-5258, doi:10.1175/JCLI-D-13-00296.1, 2014.

Rypdal, K., Rypdal, M., and H.-B. Fredriksen: Spatiotemporal Long-Range Persistence in Earth's Temperature Field: Analysis of Stochastic-Diffusive Energy Balance Models, J. Climate, 28, 8379- 8395, 2015, DOI: 10.1175/JCLI-D-15-0183.1.

Østvand, L., Nilsen, T., Rypdal, K., Divine, D., and Rypdal, M.: Long-range memory in internal and forced dynamics of millennium-long climate model simulations, Earth. Syst. Dynam., 5, 295-308, doi:10.5194/esd-5-295-2014, 2014.