Earth Syst. Dynam. Discuss., 5, 683–719, 2014
www.earth-syst-dynam-discuss.net/5/683/2014/
doi:10.5194/esdd-5-683-2014

Earth System Dynamics Discussions
Open Access

Definitions of climate and climate change under varying external conditions

C. Werndl

Department of Philosophy, Logic and Scientific Method, London School of Economics, Houghton Street, WC1A 2AE London, UK

Received: 15 May 2014 – Accepted: 24 May 2014 – Published: 10 June 2014

Correspondence to: C. Werndl (c.s.werndl@lse.ac.uk)

Published by Copernicus Publications on behalf of the European Geosciences Union.

Definitions of climate and climate change under varying external conditions

C. Werndl

Abstract

Commonly, definitions of climate are endorsed where the external conditions are held constant. This paper argues that these definitions risk being empirically void because in reality the external conditions vary. As a consequence, analogous definitions for varying external conditions are explored with help of the recently developed theory of non-autonomous dynamical systems, and the similarities and differences between the cases of constant and varying external conditions are discussed. It is argued that there are analogous definitions for varying external conditions which are preferable to the definitions where the external conditions are held constant. In this context, a novel definition is proposed (namely, climate as the distribution over time under a regime of varying external conditions), which is argued to be promising.

1 Introduction

Defining climate and climate change is highly nontrivial, as also emphasised by Todorov (1986, p. 259):

> The question of climatic change is perhaps the most complex and controversial in the entire science of meteorology. No strict criteria exist on how many dry years should occur to justify the use of the words “climatic change”. There is no unanimous opinion and agreement among climatologists on the definition of the term climate, let alone climatic change, climatic trend or fluctuation.

Commonly, definitions of climate are endorsed where the *external conditions* (i.e. what is not described by the climate variables such as solar radiation) are assumed to be *constant*. Because of the constant external conditions, one is in the realm of classical *autonomous* dynamical systems theory, and this theory has been widely used to mathematically analyse these definitions of climate (e.g. Lorenz, 1995). However, in reality the external conditions *vary*. For example, the radiation of the sun fluctuates over

Discussion Paper | Discussion Paper | Discussion Paper | Discussion Paper |

ESDD

5, 683–719, 2014

Definitions of climate and climate change under varying external conditions

C. Werndl

decades; and there are also fluctuations over short time scales such as seasonality (i.e. the seasonal variation of the solar forcing). Taking this into account leads to different definitions of climate. In particular, the recently developed theory of *non-autonomous* dynamical systems is needed to mathematically analyse them. Yet only rarely is this theory applied to analyse the notion of climate. Also, there is little discussion of the differences and similarities between the definitions for constant and varying external conditions in the climate science literature.[1]

This paper aims to contribute to fill these gaps. First, autonomous and non-autonomous dynamical systems will be introduced (Sect. 2). There are two main kinds of definitions of climate discussed in the literature: distributions over time and ensemble distributions of the possible states of the climate variables. Commonly, versions of these definitions are endorsed where the external conditions are held constant, and these definitions will be first presented. Then possible analogous definitions for varying external conditions will be explored with help of non-autonomous dynamical systems theory, and the differences and similarities between the cases of constant and varying external conditions will be discussed. The various definitions will be assessed, and it will be argued that there are definitions for varying external conditions which are preferable to the definitions for constant external conditions (Sects. 3 and 4). In this context, a *novel* definition of climate which has not been explicitly discussed in the literature will be proposed (climate as the distribution over time for a regime of varying external conditions), which will be argued to be a promising definition. Finally, the conclusion will summarize the findings (Sect. 5).

[1] In general, there has been little thorough conceptual discussion of the benefits and problems of different definitions of climate. Lorenz (1995) is a classic on this topic, but his discussion does not take into account recent developments about defining climate and about using non-autonomous dynamical systems theory to analyse climate. Werndl (2014) provides a philosophical discussion of definitions of climate. Yet because her paper is published in a philosophy journal, the discussion remains at an informal level and does not engage with the technical aspects of defining climate and climate change.

ESDD
5, 683–719, 2014

2 Non-autonomous and autonomous deterministic systems

Talk about the climate of a certain region concerns the *distribution of certain variables* (called the *climate variables*) of that region. "Region" is broadly understood (e.g., it could be London (site-specific climate) or the Earth (spatially aggregated global climate)). One often reads that climate is the expected weather (e.g. Allen, 2003; Lorenz, 1995), suggesting that *only* the variables that describe the state of the atmosphere (the *dynamical meteorological variables*) are of interest. Yet, usually the ocean variables and sometimes also other variables such as those describing ice sheets are also regarded as climate variables. What is certain is that the dynamical meteorological variables are of interest and other variables such as those describing the exact location of every animal on Earth are *not* of interest. Aside from this, there is a middle ground of variables which one might regard as climate variables for certain purposes.

There exists a huge variety of models of the evolution of the climate variables (Parker, 2006). Theoretically, climate models are conceived as models arising from differential equations, where the time parameter t is *continuous*, i.e. $t \in \mathbb{R}$ (cf. Lorenz, 1995; Palmer, 1999). For practical calculations time has to be discretized, and hence the models used in climate simulations are *discrete*, i.e. $t \in \mathbb{Z}$ or $\mathbb{N}_0$ (the dynamics is called *non-invertible* when $t \in \mathbb{N}_0$). This paper will adopt the framework of discrete time, but all the results carry over to continuous time.

Suppose that a climate model is given where the external conditions are held *constant*. Then one obtains an autonomous deterministic system (analysable by classical autonomous dynamical systems theory). More specifically, an *autonomous deterministic system* (X, Σ_X, T) consists of a set X (the *phase space*) representing all possible values of the climate variables, a σ-algebra Σ_X representing all subsets of X of interest, and the measurable functions $T(x,t) : X \times \mathbb{Z} \to X$ or $T(x,t) : X \times \mathbb{N}_0 \to X$ (the *evolution equations*). The function $T_x(t) := T(x,t)$, for a certain fixed initial value of the climate variables x, is called the *solution* and gives one the value of the climate variables after t time steps. Suppose that $T(x,t)$ is bijective for all x and t and there is a probability

ESDD
5, 683–719, 2014

measure μ on X which is invariant, i.e. $\mu(T(A,t)) = \mu(A)$ for all $A \in \Sigma_X$ and all t. Then $(X, \Sigma_X, T(x,t), \mu)$ is a *measure-preserving deterministic model*.

In reality the external conditions vary. Hence realistic models of the evolution of the climate variables are non-autonomous deterministic systems. The recently developed theory of non-autonomous dynamical systems is needed to analyse them. More specifically, for a *non-autonomous deterministic system* $(X, \Sigma_X, T(x,t_0,t))$ the evolution equations are a family of measurable functions $T(x,t_0,t) : X \times \mathbb{Z} \times \mathbb{Z} \rightarrow X$ or $T(x,t_0,t) : X \times \mathbb{N}_0 \times \mathbb{N}_0 \rightarrow X$ that *depend* on the time point t_0 (X and Σ_X are defined as before). The *solution* when x is the initial value of the climate variables at time t_0 is the function $T_{x,t_0}(t)$ and gives one the value of the climate variables at time t (cf. Kloeden and Rasmussen, 2011). (Next to deterministic models sometimes also stochastic models are used. For all the definitions of climate discussed in this paper there are corresponding stochastic definitions.)

Autonomous and non-autonomous deterministic models constitute an extremely general class of models. Measure-preserving deterministic models are a more specific class, including Axiom A systems, which are potentially relevant to climate science. A set $\Delta \subseteq X$ (X a smooth manifold) is *hyperbolic* iff (if and only if) its tangent bundle is split into three invariant subspaces, one exponentially contracting, one exponentially expanding, and one 1-dimensional space tangential to the flow direction. A point x in X is wandering iff there is an open set B containing x such that $B \cap T(B,t) \neq \emptyset$ for all sufficiently large t. The *nonwandering* set of X consists of those points in X which are not wandering. Now $(X, \Sigma_X, T(x,t), \mu)$ is an *Axiom A* system if its nonwandering set is hyperbolic and the periodic orbits are dense in X. We also assume, as often done, that an Axiom A system is *ergodic*, i.e. for all $A \in \Sigma_X$

$$\lim_{k\to\infty} \frac{1}{k} \sum_{i=0}^{k-1} \chi_A(T_t(x)) = \mu(A), \tag{1}$$

for any $x \in B$ with $\mu(B) = 1$ (where χ_A is the characteristic function of A, i.e. $\chi_A(x) = 1$ for $x \in A$ and 0 otherwise) (cf. Petersen, 1983). The measures μ of Axiom A systems

Discussion Paper | Discussion Paper | Discussion Paper | Discussion Paper |

ESDD

5, 683–719, 2014

Definitions of climate and climate change under varying external conditions

C. Werndl

are *SBR-measures*, i.e. they are measures that have absolutely continuous conditional measures along unstable manifolds (Eckmann and Ruelle, 1985; Young, 2002). According to the chaotic hypothesis by Gallavotti and Cohen (1995), many degree of freedom nonequilibrium systems behave essentially like Axiom A systems for the purpose of computing macroscopic properties. This implies that Axiom A systems could be relevant to climate science.

This paper will present simulations of a simple example to illustrate the various definitions of climate. More specifically, let us assume that the only climate variable is the *temperature* (with possible values in [0,30]), and that its evolution is given by:

$$x_{t+1} = f(x_t) = \alpha_t x_t \left(1 - \frac{x_t}{30}\right), \tag{2}$$

where x_t denotes the temperature at day t. Here the external conditions consist just of α_t, which represents the solar energy reaching the Earth at day t. α_t is assumed to be periodically fluctuating between the values 3.2 and 4 (and $\alpha_t = 3.2$ on 1 July 1984). In Fig. 1 the evolution equation is shown when $\alpha_t = 3.2$ (left) and $\alpha_t = 4$ (right): note that for constant α_t this equation amounts to the logistic map. The logistic map (with constant $\alpha_t = 4$) was also investigated by Lorenz (1964, p. 3) as the "simplest possible system of nonlinear difference equations, among those systems capable of generating a stable climate".

3 Climate as distribution over time

3.1 Concept T1. Distribution over time under constant external conditions

The idea of climate as a distribution over time is very popular. In particular, a widely endorsed concept is that climate is the *distribution over time which arises under constant external conditions*. Two definitions are obtained by either considering a *finite*

ESDD
5, 683–719, 2014

or *infinite* time period (both are popular, e.g., Dymnikov and Gritsoun, 2001; Lorenz, 1995; Palmer, 1999).

More specifically, let $(X, \Sigma_X, T(x,t))$ be the true deterministic model of the evolution of the climate variables where the external conditions are held constant at a certain value (such models are analysed with autonomous dynamical systems theory). The finite distribution is given by $T_x(t)$ for $0 \leq t \leq k-1$, where x is the initial value of the climate variables and the concern are k time steps. That is, a set $A \in \Sigma_X$ is assigned the measure

$$\frac{1}{k}\sum_{t=0}^{k-1} \chi_A(T_x(t)), \tag{3}$$

where χ_A is the characteristic function of A. Hence the value of a *macroscopic observable*, i.e. a square integrable function $f: X \to \mathbb{R}$, is $\frac{1}{k}\sum_{t=0}^{k-1} f(T_x(t))$. The infinite distribution is given by $T_x(t)$ for $t \geq 0$. That is, a set $A \in \Sigma_X$ is assigned the measure (these limits usually exist):

$$\lim_{k\to\infty} \frac{1}{k}\sum_{t=0}^{k-1} \chi_A(T_x(t)), \tag{4}$$

and the value of the macroscopic observable f is $\lim_{k\to\infty} \frac{1}{k}\sum_{t=0}^{k-1} f(T_x(t))$.

In reality the external conditions are not constant. Suppose that the external conditions take the form of small fluctuations around a mean value c over a certain time period. Then the climate over this period is defined as the (finite or infinite) distribution that arises under c (for the infinite version it is assumed that the distributions over longer time periods of interest are approximated by the infinite distribution). When there are different distributions the climate differs, and different distributions for two successive time periods amount to *climate change*. It can arise because of different external conditions (external climate change) or because of different initial values for constant

ESDD
5, 683–719, 2014

external conditions (internal climate change). When there is a change in the external conditions, natural variability (i.e. changes from internal interactions between components of the climate system) can blur the detection of this change.

To illustrate this concept, consider our simple example of the evolution of the temperature (cf. Sect. 2). Suppose that the initial temperature was 25.25 °C on 1 July 1984. The amount of solar energy that reaches the Earth fluctuated around the mean value $\alpha = 3.6$ from 1 July 1984 to 30 June 2014. Hence the climate (finite version) is given by the finite distribution for constant $\alpha = 3.6$ over the period from 1 July 1984 to 30 June 2014 (Eq. 3). This finite distribution is shown in Fig. 2. The infinite version of climate is given by the infinite distribution, which arises by letting k go to infinity (Eq. 4). In simulations this infinite distribution essentially looks like a smoother version of Fig. 2, in line with what we know from theoretical results (cf. Jacobsen, 1981; Lyubich, 2002).

This concept of climate refers to distributions of the true climate model under constant external conditions. The problem is that such distributions may not relate to the past and future distributions of the actual climate system *where the external conditions vary*. More specifically, first, there are time periods where the external conditions are not small fluctuations around a mean value c but vary considerably (this is a common situation for climate predictions), and then the concept is not applicable. Second, suppose that the external conditions indeed take the form of small fluctuations around c. Then proponents of this concept assume that the distributions of the actual climate system are approximately equal to the distributions under constant external conditions c. However, *this assumption is doubtful*.

This can again be illustrated with our simple climate model. The actual finite distribution over time is shown in Fig. 3. It is the distribution arising for the actual fluctuations of the solar energy α_t between 3.2 and 4 from 1 July 1984 to 30 June 2014 (with $\alpha_t = 3.2$ on 1 July 1984 and with initial temperature 25.25). The actual infinite distribution arises by taking the limit of this distribution for k to infinity (in simulations this infinite distribution looks like a smoother version of Fig. 3). Obviously, the temperature distributions for constant external conditions $\alpha = 3.6$ (Fig. 2) and for the actual external conditions

ESDD
5, 683–719, 2014

(Fig. 3) are very different (the p value for a two sample Kolmogorov–Smirnov test is approximately 6.3793×10^{-580}).

Other studies have come to similar conclusions. For instance, Daron (2012) numerically investigated the Lorenz (1963) system when one parameter is subject to non-periodic fluctuations. He found that the (finite and infinite) distributions can differ significantly from the (finite and infinite) distributions arising when the parameters are held fixed. A resonance effect, which can also arise for small fluctuations, is responsible for the different distributions. Furthermore, even when disregarding longer-term fluctuations, there are short-term fluctuations such as seasonality. A growing body of work indicates that seasonality leads to different distributions over time. In particular, seasonality is expected to lead to an increase of the average surface temperature. More specifically, it was found in Goswami et al. (2006) for a model of the monsoon, Jin et al. (1994) for a model of the El Niño, Kurgansky et al. (1996) for a baroclinic low-order model of the atmosphere and Lorenz (1990) for a simple general circulation model that the (finite and infinite) distributions over time are different when seasonality is included. Similar results are likely to hold for the true climate model. Thus this concept may be *empirically void*, showing a need to take the varying external conditions into account when defining climate. So one wonders whether a similar definition can be found where the external conditions are allowed to vary.

3.2 Concept T2. Distribution over time for the actual path of the external conditions

The most direct way to achieve this is to define climate as the *distribution over time of the actual climate system* (i.e. given the actual path of the external conditions). Again, two definitions are obtained by considering a *finite* or an *infinite* time period.

More specifically, let $(X, \Sigma_X, T(x, t_0, t))$ be the true deterministic model of the evolution of the climate variables where the external conditions vary as is in reality (the recently developed theory of non-autonomous dynamical systems is needed to analyse such models). Then climate is the distribution of $T_{x,t_0}(t)$ for $t_0 \leq t \leq t_1 - 1$ (when the time steps

Discussion Paper | Discussion Paper | Discussion Paper | Discussion Paper |
ESDD
5, 683–719, 2014

from t_0 to $t_1 - 1$ are of concern), where x is the initial value of the climate variables at t_0. That is, the measure of $A \in \Sigma_X$ is

$$\frac{1}{t_1 - t_0} \sum_{t=t_0}^{t_1-1} \chi_A(T_{x,t_0}(t)), \tag{5}$$

and the value of the macroscopic observable f is $\frac{1}{t_1-t_0} \sum_{t=t_0}^{t_1-1} f(T_{x,t_0}(t))$.

In the infinite case *because the equations are non-autonomous, there are two possible limits* that could define the climate (they often differ). Namely, one possibility is to let t_1 go to infinity and to assign to $A \in \Sigma_X$ the measure

$$\lim_{t_1\to\infty} \frac{1}{t_1 - t_0} \sum_{t=t_0}^{t_1-1} \chi_A(T_{x,t_0}(t)), \tag{6}$$

and then the value of the macroscopic observable f is $\lim_{t_1\to\infty} \frac{1}{t_1-t_0} \sum_{t=t_0}^{t_1-1} f(T_{x,t_0}(t))$. The other possibility is to let t_0 go to minus infinity and to assign to $A \in \Sigma_X$ the measure

$$\lim_{t_0\to-\infty} \frac{1}{t_1 - t_0} \sum_{t=t_0}^{t_1-1} \chi_A(T_{x,t_0}(t)), \tag{7}$$

and then the value of the macroscopic observable f is $\lim_{t_0\to-\infty} \frac{1}{t_1-t_0} \sum_{t=t_0}^{t_1-1} f(T_{x,t_0}(t))$. The limits Eqs. (6) and (7) need not exist, and it is unclear whether they exist for realistic climate models (Mancho et al., 2013).

Let us illustrate this concept again with our simple model of the temperature evolution (cf. Sect. 2). In the finite case the climate from 1 July 1984 to 30 June 2014 is the temperature distribution shown in Fig. 3. The infinite distributions arises for $t_1 \to \infty$ or $t_0 \to -\infty$. In simulations both infinite distributions simply look like a smoother version of

ESDD

5, 683–719, 2014

Title Page
Abstract | Introduction
Conclusions | References
Tables | Figures
◀| |▶
◀ ▶
Back | Close
Full Screen / Esc
Printer-friendly Version
Interactive Discussion

Fig. 3. Thus this seems to be one of the rare cases where the two infinite distributions agree (this is explained by the periodically fluctuating external conditions).

The infinite version is usually deemed to be *unacceptable* since, by definition, there *cannot be climate change* (cf. Lorenz, 1995). The finite version is very popular: because climate simply is a distribution of the actual climate system over a certain finite time period, one cannot imagine a definition of climate that is easier to estimate from the observations. In particular, *standard climate normals* are statistics taken over a period of thirty years published by the World Meteorological Organisation (e.g., from 1961 to 1990). Thus the climate is often taken to be the actual distribution of the climate system over thirty years (cf. Hulme et al., 2009).

However, for the finite version there is also a serious problem involving climate change. As before, it is natural to say that different distributions correspond to different climates and that climate change amounts to different distributions for succeeding time periods. Again, there can be external and internal climate change (and natural variability can blur the detection of a change in the external conditions). Now consider a scenario where the climate is defined over a time period from t_0 to $t_1 - 1$. Suppose that this time period is marked by two different regimes because at $t_m = t_0 + \frac{t_1 - 1 - t_0}{2}$ the Earth was hit by a meteor and thus became a much colder place. Clearly, "the" climate over the time period from t_0 to $t_1 - 1$ does not exist. Instead, one finds two climates over this time period: one before t_m and a different one after t_m. However, the second concept does *not* imply this as the climate simply amounts to the distribution from t_0 to $t_1 - 1$! That is, the external conditions can change drastically over the time period over which the climate distribution is defined. This concept has *no* means to require that such a regime change cannot occur because all it says is that climate is a distribution arising over a certain time period. Therefore, it leads to wrong judgements about climate change.

ESDD
5, 683–719, 2014

3.3 Concept T3. Distribution over time relative to a regime of varying external conditions

Another definition will now be introduced which avoids these problems. Namely, let climate be defined as the *distribution over time under a certain regime of varying external conditions*. Some thoughts need to be given on what should count as a regime of varying external conditions (and this is certainly no easy question). For example, it is reasonable to require that the mean of the external conditions should be (approximately) constant. Again, one obtains two versions by either considering a *finite* or *infinite* time period. This definition is novel in the sense that it has not been explicitly discussed as a definition of climate in the literature.

More specifically, consider the true model of the evolution of the climate variables $(X, \Sigma_X, T(x, t_0, t))$ subject to a certain regime of varying external conditions (non-autonomous dynamical systems theory is needed to analyse such models). Equation (5) gives the finite distribution of $T_{x,t_0}(t)$ for $t_0 \le t \le t_1 - 1$ (when the time period from t_0 to $t_1 - 1$ is of concern), where x is the initial value of the climate variables at t_0. Because the equations are *non-autonomous, there are two possible infinite distributions*: one for $t_1 \to \infty$ (Eq. 6) and one for $t_0 \to -\infty$ (Eq. 7). Note that all these distributions are different from the ones of the second concept because the external conditions are subject to a certain regime. As before, it is unclear whether the limits Eqs. (6) and (7) exist for the true or realistic climate models.

Suppose that over a time period the actual external conditions vary according to a certain regime. Then the climate over this time period is given by the finite distribution (for the finite version) or by one of the infinite distributions (for the infinite version; here it is assumed that the distribution over longer time periods of interest are approximated by the infinite distribution). Climate is a distribution of the true climate model under a certain regime of varying external conditions. In the finite case this distribution will often coincide with a distribution over time of the actual climate system (when the actual climate system is subject to the regime of varying external conditions from t_0 to

ESDD
5, 683–719, 2014

$t_1 - 1$). Hence for the finite version there is an immediate link to the observations. Different distributions correspond to different climates. *Climate change* amounts to different climates over two successive time periods, and there can be external and internal climate change (and, again, natural variability can blur the detection of a change in the external conditions). This concept avoids the problems with conceptualising climate change encountered by the second concept: e.g., the conditions before and after the meteor hit the Earth correspond to different regimes of external conditions with a different climate.

For our simple climate model (cf. Sect. 2) a possible regime of varying external conditions is when the solar energy fluctuates periodically between 3.2 and 4. Note that α_t was subject to this regime from 1 July 1984 to 30 June 2014. Consequently, in the finite case, the climate is the distribution from 1 July 1984 to 30 June 2014 under this regime of varying external conditions (with initial temperature 25.25) as shown in Fig. 3. In the infinite case climate is the distribution that arises when $t_1 \to \infty$ or when $t_0 \to -\infty$ for the regime of the periodically fluctuating solar energy. As already mentioned, in simulations both infinite distributions simply look like a smoother version of Fig. 3.

Let us now analyse the main differences and similarities between the cases of constant (concept T1) and varying external conditions (concept T2 and T3) and assess these definitions.

3.4 Constant vs. varying external conditions: differences, similarities and assessment

For any definition of climate as distribution over time (including concepts T1–T3) the climate *depends on the initial value* of the climate variables x. This does not constitute a problem, but there is an important difference. For *constant external conditions* (concept T1) two results are known under which conditions the infinite climate distribution is *independent of the initial conditions* (for the finite version no such results exist). Yet there are *no known corresponding results for varying external conditions (concepts T2 and T3)*.

ESDD
5, 683–719, 2014

Let me state the two independence results for constant external conditions. First, if the system is ergodic (Eq. 1), then it immediately follows that for almost all initial values the measure assigned to a set A by the infinite distribution (Eq. 4) is the same (namely, $\mu(A)$). Second, an attractor Ω, $\Omega \subseteq X \subseteq \mathbb{R}^n$, is an invariant set which attracts all initial values in the *basin of attraction* X, i.e., $\lim_{t\to\infty} \mathrm{dist}(T_t(x), \Omega) = 0$ for all $x \in X$. Now a physical measure is approximated by the distributions taken over time for initial values in the basin of attraction. Formally: μ_Ω is a *physical measure* on the attractor Ω if for Lebesgue-almost all $x \in X$

$$\lim_{k\to\infty} \frac{1}{k} \sum_{t=0}^{k-1} \chi_A(T_t(x)) = \mu_\Omega(A), \tag{8}$$

whenever $\mu_\Omega(\delta A) = 0$ (δA is the boundary of A) (cf. Eckmann and Ruelle, 1985; Ruelle, 1976). Hence for a physical measure the measure assigned to any $A \in \Sigma_X$ with $\mu_\Omega(\delta A) = 0$ by the infinite distribution (Eq. 4) is the same for Lebesgue-almost all initial values.

For instance, theoretical results suggest that our simple example of the logistic map with constant $\alpha = 3.6$ is ergodic (Jacobsen, 1981; Lyubich, 2002). Therefore, the climate as the infinite distribution is the same for Lebesgue-almost all initial values. Axiom A systems are ergodic and Axiom A attractors (i.e., when the motion on the attractor is an Axiom A system) have physical measures because they have SBR-measures. While these systems are potentially relevant to climate science, there is uncertainty whether the true or realistic climate models under constant external conditions are ergodic or have physical measures (Lorenz, 1976; McGuffie and Henderson-Sellars, 2005; Schneider and Dickinson, 2000). Therefore, also for concept T1 one has to be vary of assuming that the infinite climate distribution is independent of the initial value. Still, there is the important difference that while for constant external conditions (concept T1) general independence results are known, this is not the case for varying external conditions (concepts T2 and T3).

Discussion Paper | Discussion Paper | Discussion Paper | Discussion Paper |

ESDD
5, 683–719, 2014

Title Page
Abstract | Introduction
Conclusions | References
Tables | Figures
◀| |▶
◀ ▶
Back | Close
Full Screen / Esc
Printer-friendly Version
Interactive Discussion

Another difference is that for varying external conditions there are often two different infinite distributions. Hence *the problem arises which of them is the climate*. If the concern is the past, one can check which infinite distribution is approximated by the observed distributions and identify it with the climate. Yet if the concern is the future, it is unclear how to make a choice.

Let us now turn to the similarities. For any definition of climate as finite distribution over time (including the finite versions of concepts T1–T3) the question arises *over which finite time period the distributions should be taken*. In the climate system processes are happening at various different time scales, and even when just looking at a component of the climate system such as the ocean or the biosphere various different time scales are present (Solomon et al., 2007). Because of these various time scales, selecting a standard period for all practical purposes does not seem feasible. Instead, a pragmatic perspective as outlined in Lorenz (1995) seems promising: the choice of the time interval should be influenced by the purpose of research, e.g., if the interest is inter-glacial climate, the time period will be relatively long. What is important, though, is that the time period should be short enough to avoid that changes conceived as climatic are subsumed under the same climate but long enough so that no specific predictions can be made (e.g., longer than the predictability horizon given by the El Niño). Note also that there will nearly always be climate change. Yet this does not constitute a problem: one can say that all that matters is when the distributions differ significantly.

For any definition of climate as infinite distribution over time (including the infinite versions of concepts T1–T3) there is the advantage that *infinite distributions are easier to deal with mathematically* than finite distributions. However, there is also the *worry whether the finite distributions over longer time periods of interest are approximated by the the infinite distributions*. If this is not the case, the infinite distributions are empirically void. For several climate models that incorporate some realism about the climate system (including models with constant and varying external conditions) distributions taken over long finite time periods differ from the distributions in the infinite limit (Bhattacharya et al., 1982; Daron, 2012; Sempf et al., 2007; Daron and Stainforth, 2013). It

ESDD
5, 683–719, 2014

remains an open question whether similar results hold for the true or realistic climate models. Yet there remains the *worry that the infinite distributions are empirically void.*[2]

In sum, what emerges from the discussion of climate as distribution over time (concepts T1–T3) is as follows. Because in reality the external conditions vary (both on short and long time scales), concept T1 where the external conditions are assumed to be constant risks being empirically void. Instead, a definition for varying external conditions such as concept T2 or T3 is needed. Concept T2 fails to conceptualise climate change in a satisfactory way. Furthermore, distributions over finite time periods are preferable to distributions over an infinite time period because the latter may refer to mathematical limits which do not exist, risk being empirically void, and (for varying external conditions) there is the problem which of the two infinite distributions should be identified with the climate. *Consequently, the finite version of concept T3 (climate as finite distribution under a certain regime of varying external conditions) is the most promising definition of climate as a distribution over time.*

4 Climate as ensemble distribution

4.1 Concept E1. Ensemble distribution for constant external conditions

Another popular idea is that climate is an ensemble distribution, i.e., a distribution of possible values of the climate variables (ensemble distributions are altogether different from distributions over time). In particular, climate is often identified with the *future ensemble distribution conditional on our uncertainty in the initial values under constant external conditions*. There is again a *finite* and an *infinite* version (e.g., Lorenz, 1995; Stone and Knutti, 2010; Stone et al., 2009).

[2]In the case of constant external conditions the notion of *almost intransitivity* was introduced by Lorenz to characterise systems where distributions over long finite time periods differ from one time period to the next and thus also from the infinite distributions. Lorenz (1968, 1970, 1976, 1995) believed that realistic climate systems may well be almost intransitive.

ESDD
5, 683–719, 2014

More specifically, suppose that the external conditions take the form of small fluctuations around a mean value c over a certain time period. Suppose that the concern is to make predictions k time steps into the future, and denote by p_0 the probability density describing the present uncertainty about the initial values of the climate variables. Then for the finite version the climate after k time steps is defined as the distribution arising when p_0 is evolved k time steps forward under the true climate model with *constant external conditions* c (classical autonomous dynamical systems theory can be used to analyse such a model $(X, \Sigma_X, T(x,t))$). That is, the measure of $A \in \Sigma_X$ is:

$$\int_A p_k \mathrm{d}\mu, \tag{9}$$

(p_k denotes the probability density p_0 evolved k time steps forward), and the value of the macroscopic observable f is $\int_X f p_k \mathrm{d}\mu$. For the infinite version the climate after k time steps is the distribution of the possible values of the climate variables as time goes to infinity, i.e. the measure of $A \in \Sigma_X$ is:

$$\lim_{k\to\infty} \int_A p_k \mathrm{d}\mu, \tag{10}$$

and the value of the macroscopic observable f is $\lim_{k\to\infty} \int_X f p_k \mathrm{d}\mu$. The limits Eq. (10) may not exist and it is unclear whether they exist for the true or realistic climate models (Lasota and Mackey, 1985; Provatas and Mackey, 1991). Again, different distributions correspond to different climates.

To illustrate this definition, consider again our simple climate model of the evolution of the temperature (cf. Sect. 2). Suppose that the temperature was found to be between 25.00 °C and 25.10 °C on 1 July 2014. Let p_0 be the uniform probability density over [25.00, 25.10] representing this measurement. Suppose further that the aim is to predict the temperature value at t_1 = 1 July 2050. The amount of solar energy that reaches the Earth will fluctuate around $c = 3.6$ from 1 July 2014 to 1 July 2050. The climate on

ESDD
5, 683–719, 2014

1 July 2050 (the finite version for constant $c = 3.6$, i.e., Eq. 9) is the distribution shown in Fig. 4. The infinite version of this concept of climate is not well defined because the limit Eq. (10) does not exist (there is no convergence in the simulations, in line with theoretical results suggesting that ensembles fluctuate periodically between different regions of phase space – cf. Jacobsen, 1981; Lyubich, 2002).

This concept refers to ensembles of the true climate model under *constant* external conditions. The problem is that such ensembles may be useless for predicting the actual climate system *where the external conditions vary*. More specifically, often one wants to make predictions when the external conditions vary considerably (this is a common situation for climate predictions), but then the concept is not applicable. Second, suppose that the external conditions indeed take the form of small fluctuations around c. Then proponents of this concept assume that the ensemble distributions of the actual climate system approximately equal the ensemble distributions under constant external conditions c. However, *this assumption is doubtful*.

This is again illustrated by our simple climate model. The actual ensemble distribution on 1 July 2050 is shown in Fig. 5. This is the distribution which arises when the density p_0 is evolved forward to 1 July 2050 under the periodically fluctuating external conditions (again, the infinite limit of this distribution does not seem to exist). Obviously, the finite version of the ensemble definition of climate for constant external conditions $c = 3.6$ (Fig. 4) and the actual ensemble distribution (Fig. 5) are very different (the p value for a two sample Kolmogorov–Smirnov test is approximately $1.58996 \times 10^{-3981}$).

Studies of climate systems have led to similar conclusions. For example, Daron (2012) numerically investigated the Lorenz (1963) system model when one parameter is subjected to aperiodic fluctuations. He found that the (finite and infinite) ensemble distributions can differ significantly from the (finite and infinite) ensemble distributions under constant external conditions. A resonance effect, which can also arise for small fluctuations, is responsible for the different distributions. Further, for the climate system, even when disregarding longer-term fluctuations, there are short-term fluctuations such as seasonality. It was found in Gowsami et al. (2006) for a model of the monsoon, Jin

ESDD
5, 683–719, 2014

et al. (1994) for a model of the El Niño and Lorenz (1990) for a very simple general circulation mode that the ensemble distributions differ when seasonality is included (in particular, seasonality leads to an increase of the average surface temperature). Similar results are likely to hold for the true climate model. *Thus there is the worry that this concept is empirically void*, showing a need to take the varying external conditions into account.

4.2 Concept E2. Ensemble distribution for the actual path of the external conditions

This can be achieved by defining climate as the *future ensemble distribution conditional on our uncertainty in the initial values when the external conditions vary as in reality*. Again a *finite* and an *infinite* version are endorsed (e.g., Checkroun et al., 2011; Daron, 2012; Daron and Stainforth, 2013; Smith, 2002). Because the external conditions vary as in reality, particularly the finite version is very attractive for predictive purposes because it quantifies the likelihood of the future possible properties of the climate system relative to the present uncertainty.

More specifically, let the true model $(X, \Sigma_X, T(x, t_0, t))$ of the evolution of the climate variables be given (non-autonomous dynamical systems is needed to describe such models). Denote by p_{t_0} the probability density on X that describes the uncertainty about the climate variables at t_0. Further, suppose that the concern is to make predictions at time t_1 in the future. In the finite case the climate at t_1 is defined as the distribution arising when p_{t_0} is evolved to t_1, i.e. the measure of $A \in \Sigma_X$ is:

$$\int_A p_{t_0,t_1} \mathrm{d}\mu \tag{11}$$

(p_{t_0,t_1} is the probability density p_{t_0} evolved forward to t_1). Then the value of the macroscopic observable f is $\int_X f p_{t_0,t_1} \mathrm{d}\mu$. *Because the equations depend on time, in the infinite case there are two possibilities*. The climate at t_1 can either be defined by letting

ESDD
5, 683–719, 2014

t_1 go to infinity and assigning to $A \in \Sigma_X$ the measure

$$\lim_{t_1 \to \infty} \int_A p_{t_0,t_1} \mathrm{d}\mu, \tag{12}$$

and then the value of the macroscopic observable f is $\lim_{t_1 \to \infty} \int_X f p_{t_0,t_1} \mathrm{d}\mu$. Another possibility is to let t_0 go to minus infinity and to assign to $A \in \Sigma_X$ the measure

$$\lim_{t_0 \to -\infty} \int_A p_{t_0,t_1} \mathrm{d}\mu, \tag{13}$$

and then the value of the macroscopic observable f is $\lim_{t_0 \to -\infty} \int_X f p_{t_0,t_1} \mathrm{d}\mu$. The two infinite distributions usually differ. In general, the limits Eq. (12) usually do not exist. Thus the limits Eq. (13) are more relevant (but sometimes they also do not exist – Checkroun et al., 2011; Kloeden and Rasmussen, 2011).

At this point it should be mentioned that *response theory* provides a powerful framework for calculating ensemble distributions for a certain class of systems that are initially in equilibrium and then perturbed by small non-autonomous fluctuations. The value of response theory is that the ensembles can be computed in terms of expectation values of explicit and computable functions averaged over the invariant measures corresponding to the unperturbed systems (Lucarini and Sarno, 2011; Lucarini, 2012; Ruelle, 2009). Thus the ensembles can be calculated without long-run simulations and integrations of the perturbed system. Rigorously the theory has been established for Axiom A attractors. Yet the theory is expected to be of much broader applicability: e.g. Lucarini (2009) recently provided evidence that it also applies to the Lorenz (1963) system (which is not Axiom A) perturbed by a weak periodic forcing.

To come back to our simple climate model: let p_{t_0} be the uniform probability density over [25.00, 25.10], representing our uncertainty in the initial values on 1 July 2014. In the finite case the climate is the distribution shown in Fig. 5, which arises when p_{t_0} is evolved forward to 1 July 2050 (Eq. 11). In the infinite case climate could be either

Discussion Paper | Discussion Paper | Discussion Paper | Discussion Paper |

ESDD

5, 683–719, 2014

Definitions of climate and climate change under varying external conditions

C. Werndl

Title Page
Abstract Introduction
Conclusions References
Tables Figures
◀◀ ▶▶
◀ ▶
Back Close
Full Screen / Esc
Printer-friendly Version
Interactive Discussion

defined by letting t_1 go to infinity (Eq. 12) or by letting t_0 go to minus infinity (Eq. 13), but simulations suggest that none of these limits exist for our simple model.

Let us now analyse the main differences and similarities between the case of constant (concept E1) and varying external conditions (concept E2) and assess these definitions.[3]

4.3 Constant vs. varying external conditions: differences, similarities and assessment

Climate as ensemble distribution (including concept E1 and E2) has the undesirable consequence that it *depends on the initial uncertainty about the climate variables*. Some might reply to this that there is no such dependence because for the true evolution of the climate variables the memory of the initial values washes out over time. That this reply is not always adequate is illustrated by our simple climate model, where simulations show that the distributions (both under constant and varying external conditions) depend on the initial uncertainty. Still, there are *independence results for the infinite versions of climate (both for constant and varying external conditions)* (for the finite version no such results exist). The results for varying external conditions are recent and not well known and thus of particular interest.

Let us first look at the case of constant external conditions. First, a measure-preserving deterministic model $(X, \Sigma_X, T(x,t), \mu)$ is *mixing* iff for all densities p_0 and all $A \in \Sigma_X$ (Berger, 2001; Lasota and MacKey, 1985):

$$\lim_{k\to\infty} \int_A p_k \mathrm{d}\mu = \mu(A). \tag{14}$$

[3]It would be possible to introduce an ensemble definition for *regimes of varying external conditions* (corresponding to Concept T3). Definition T3 was introduced because it avoids a serious problem of Concept T2 (that different climates are not classified correctly). By introducing an ensemble definition for regimes of varying external conditions no problems are avoided (in comparison to Definition E2). Hence no need arises for such a concept.

ESDD
5, 683–719, 2014

It follows immediately from this condition that the measure assigned to a set by equation (10) is the same for all initial densities p_0 and hence there is no dependence on the uncertainty.[4]

The second result is about measures that are approximated by the distribution of initial densities on the basin of attraction that are evolved forward (called strong physical measures). Formally, μ^{Ω} on the attractor Ω is called a *strong physical measure* iff for any density p_0 (relative to the Lebesgue measure λ) on X and any $A \in \Sigma_X$:

$$\lim_{k\to\infty} \int_A p_k \mathrm{d}\lambda = \mu^{\Omega}(A), \tag{15}$$

whenever $\mu^{\Omega}(\delta A) = 0$ (δA denotes the boundary of A) (cf. Ruelle, 1976; Tasaki et al., 1998). Strong physical measures are physical (but the converse does not hold). It immediately follows that the dependence on the uncertainty is negligible in the sense that the measure assigned to any set A with $\mu^{\Omega}(\delta A) = 0$ by Eq. (10) is the same for all initial uncertainties.

For varying external conditions there exists a result corresponding to the second one. For this we first have to introduce pullback attractors (a very useful notion of attractor for non-autonomous dynamics). In the non-autonomous case a set $\Omega \subseteq \mathbb{R} \times \mathbb{R}^n$ is *invariant* iff $\Omega(t) = T(\Omega(t_0), t_0, t)$ for all $t, t_0 \in \mathbb{R}$, where $\Omega(t) := \{x \in \mathbb{R}^n \mid (t, x) \in \Omega\}$. A *pullback attractor* $\Omega \subseteq \mathbb{R} \times \mathbb{R}^n$ is an invariant set where for all initial values $x \in X$

$$\lim_{t_0\to-\infty} \mathrm{dist}(T(x, t_0, t), \Omega(t)) = 0. \tag{16}$$

[4] When does the stronger condition hold that any initial probability density p_0 converges to the measure μ, i.e. $\lim_{k\to\infty} \int_X |1 - p_k| \mathrm{d}\mu = 0$? Most climate models are *invertible* (i.e. $t \in \mathbb{R}$ or $\mathbb{Z}$). Then this stronger condition *cannot* hold because of the invariance of the measure. For *non-invertible* models $(X, \Sigma_X, T(x,t), \mu)$ this condition holds iff they are *exact*, i.e. when $\lim_{t\to\infty} \mu(T(A,t)) = 1$ for all $A \in \Sigma_X$, $\mu(A) > 0$ (Berger, 2001; Lasota and Mackey, 1985). Exactness is an even stronger condition than mixing, and whether realistic non-invertible climate models are exact is unknown.

ESDD
5, 683–719, 2014

Now, non-autonomous strong physical measures μ_t^{Ω} defined on a pullback attractor $\Omega(t)$, $t \in \mathbb{R}$, are approximated by the distribution of initial densities in the basin of attraction that are evolved forward to t (note that the contractive nature of the dynamics is what allows for the construction of this very useful concept). They are the analogues of strong physical measures. Formally: μ_t^{Ω} are *non-autonomous strong physical measures* iff for all t, t_0, all densities p_0 at time t_0 (relative to the Lebesgue measure λ) on X and all $A \in \Sigma_X$ (cf. Buzzi, 1999; Checkroun et al., 2011):

$$\lim_{t_0 \to -\infty} \int_A p_{t_0,t} \mathrm{d}\lambda = \mu_t^{\Omega}(A), \tag{17}$$

whenever $\mu_t^{\Omega}(\delta A) = 0$. Therefore, the dependence on the uncertainty is negligible in the sense that the measure of any set A with $\mu_t^{\Omega}(\delta A) = 0$ is the same for all initial uncertainties.

The problem with these independence results is that the assumptions which have to be satisfied (mixing, strong physical measures, non-autonomous strong physical measures) amount to strong dynamical conditions (cf. Werndl, 2009). They are not satisfied for our simple model under constant $\alpha_t = 3.6$ or under the periodically fluctuating external conditions (cf. Sect. 2). Axiom A systems are mixing and Axiom A attractors have strong physical measures, and these systems are potentially relevant to climate science. Still, there is uncertainty whether these dynamical conditions hold for the true or realistic climate models. Hence *even for the infinite version there remains the problem that the climate may well depend on our uncertainty in the initial values*.

Another similarity is that for climate as the infinite ensemble distribution (including the infinite versions of concepts E1 and E2) there is the advantage that infinite distributions are *easier to deal with mathematically* than finite distributions. However, there is also the worry *whether the infinite ensemble distributions are approximated by the finite ensemble distributions* over longer time periods of interest (cf. Smith, 2002). Because in the climate system processes happen at various time scales, the rate of convergence

ESDD
5, 683–719, 2014

Definitions of climate and climate change under varying external conditions

C. Werndl

is expected to depend on the specific ensembles and will not be uniform (this can also be seen at a technical level because a spectral gap is missing in the climate case). Also, for several climate models (including models with constant and varying external conditions) the finite ensemble distributions are not close to the infinite ensemble distributions for prediction lead-times of interest (cf. Bhattacharya et al., 1982; Daron, 2012; Sempf et al., 2007; Daron and Stainforth, 2013). It is unknown whether these results carry over to the true or realistic climate models, but there remains the worry that the *infinite ensemble distributions are empirically void.*

Another similarity is that both concepts E1 and E2 are always presented as defining the climate in the *future*. So there is the question what the *past* and the *present* climate amount to (which are needed to define climate change). Nothing is stated explicitly in the literature about this. But it is natural to say that the climate of 1 July 2014 is the ensemble distribution that scientists on, say, 1 July 1984 would have predicted as the climate of 1 July 2014 relative to the uncertainty in the initial values and the constant external conditions on 1 July 1984 (for concept E1)/the actual path of the external conditions (for concept E2).[5] Then *climate change* is the change between the climates at two time points. There is external and internal climate change (the latter arises because of different uncertainties about the initial values or, for the finite version, because of different prediction lead-times). Again, natural variability can blur the detection of a change in external conditions. The climate 1 July 2014 is defined in the example by choosing a prediction lead-time of thirty years, but this choice seems arbitrary. One could argue that pragmatic considerations similar to those discussed in Sect. 3.4 (see the issue over which finite time period distributions should be defined) will fix a suitable prediction lead-time. However, then the same prediction lead-time has to be used when defining the future climate, which contradicts with standard practice (where the prediction lead-time is only determined by how many years scientists want to predict in the future). Consequently, *there remains the problem how to define the present and past climate (and the derivative notion of climate change)*.

[5]This has also been suggested by David Stainforth (personal communication, 2013).

ESDD
5, 683–719, 2014

Furthermore, there is the worry that climate as the ensemble distribution (i.e. concepts E1 and E2) *does not seem to have anything to do with the time series of past observations* (cf. Schneider and Dickinson, 2000). For the finite versions there are no general results that could provide a way out of this. However, importantly, as explained below, for the *infinite version for constant external conditions* (concept E1) two mathematical results are known which establish a *relation between ensembles and time series of observations*. Yet there can be *no* corresponding results for varying external conditions (concept E2).

Let us first state the two results for constant external conditions. If the measure-preserving deterministic system $(X, \Sigma_X, T(x,t), \mu)$ is mixing, Eqs. (14) and (1) imply that for all initial probability densities p_0 and all $A \in \Sigma_X$

$$\lim_{t\to\infty} \int_A p_t \mathrm{d}\mu = \mu(A) = \lim_{k\to\infty} \frac{1}{k} \sum_{t=0}^{k-1} \chi_A(T_t(x)), \quad (18)$$

for all $x \in B$ with $\mu(B) = 1$. Hence for almost all initial values the measure assigned to a set A by the infinite ensemble distribution and the infinite distribution over time is the same for almost all initial values. If $x \in B$ and distributions over finite long time periods approximate the distribution over an infinite time period, the infinite ensemble distribution can be estimated from the time series of observations.

Second, consider an attractor Ω with a strong physical measure μ^{Ω}. Then Eqs. (15) and (8) imply that for all initial densities p_0, all A with $\mu^{\Omega}(\delta A) = 0$ and Lebesgue-almost all initial values $x \in X$:

$$\lim_{t\to\infty} \int_A p_t \mathrm{d}\lambda = \mu^{\Omega}(A) = \lim_{k\to\infty} \frac{1}{k} \sum_{t=0}^{k-1} \chi_A(T_t(x)). \quad (19)$$

Thus for Lebesgue-almost all initial values the measure assigned to A with $\mu^{\Omega}(\delta A) = 0$ by the infinite ensemble distribution and the infinite distribution over time is the same.

ESDD
5, 683–719, 2014

If distributions over finite long time periods approximate the distribution over an infinite time period, the infinite ensemble distribution can be estimated from the time series of observations (for Lebesgue-almost all initial values).

As attractive as these results are, as discussed in Sect. 3.4, mixing and having a strong physical measure are strong dynamical conditions. Our simple model for constant $\alpha_t = 3.6$ does not satisfy any such condition (cf. Sect. 2). Axiom A systems are mixing and Axiom A attractors have strong physical measures and these systems are potentially relevant to climate science. Still, there is uncertainty whether these assumptions hold for the true or realistic climate models under constant external conditions. It is also uncertain whether for these climate models distributions taken over finite long time periods approximate the distributions over an infinite time period. Consequently, *it remains unclear whether climate as the infinite ensemble distribution under constant external conditions has anything to do with the past observations*.

The situation is *even worse for varying external conditions.* For constant external conditions there are at least results that relate ensemble distributions to distributions over time. *For varying external condition such results cannot exist.* The reason for this is that while the ensemble distributions defined by Eq. (13) depend on t_1 and thus usually change with time, the distributions over time defined by Eq. (7) is the same for all t_1. It follows that the infinite ensemble distribution cannot equal the infinite distribution over time because the former varies and the latter does not.[6]

Finally, another difference between the case of constant and varying external conditions is as follows. For concept E2 the climate will usually be identified with the distribution Eq. (13). In rare cases when the limit Eq. (12) also exists, there are *two infinite distributions* (unlike for concept E1 where there is only one infinite distribution). Then the *problem arises which of them is the climate*, and it is difficult to answer this question (past data will not help because, as argued, the climate cannot be estimated from past observations).

[6]Results about the infinite ensemble distributions (Eq. 12) and the infinite distributions over time (Eq. 6) are not of much interested since the limits Eq. (12) rarely exist.

ESDD
5, 683–719, 2014

C. Werndl

In sum, what emerges from the discussion of climate as an ensemble distribution (concepts E1–E2) is as follows. Because in reality the external conditions vary (both on short and long time scales), concept E1 where the external conditions are assumed to be constant risks being empirically void. Thus a concept such as concept E2 where the external conditions are allowed to vary is needed. Furthermore, distributions over finite time periods are preferable to distributions over infinite time periods because the latter may refer to mathematical limits which do not exist, risk being empirically void, and when there are two infinite distributions there is the problem which of them should be identified with the climate. *Therefore, the finite version of concept E2 (i.e., the finite ensemble distribution when the external conditions vary as in reality) is the most promising ensemble definition of climate.*

5 Conclusion

This paper was about the intricate issue of how to define climate and climate change. There are two main kinds of definitions of climate discussed in the literature: distributions over time and ensemble distributions of the possible states of the climate variables. In both cases a common assumption is that the external conditions are constant. That is, climate is often defined as the distribution over time under constant external conditions (concept T1) or as the ensemble distribution under constant external conditions (concept E1). This paper argued that such concepts risk being empirically void because for the actual climate system the external conditions vary (both on short and long time scales), and the distributions for varying external conditions may well differ from those under constant external conditions.

Thus there is a need for definitions which take the varying external conditions into account. A possible alternative is to define climate as the distribution over time for the actual path of the external conditions (concept T2). However, it was shown that this concept encounters serious problems in conceptualising climate change. As a response, a novel concept was put forward, namely the idea that climate is the distribution over

ESDD
5, 683–719, 2014

time under a certain regime of varying external conditions (concept T3). It was argued that this is the most promising concept of climate as a distribution over time. Then the concept of climate as the future ensemble distribution when the external conditions vary as in reality (concept E2) was discussed, and it was claimed that this is the most promising concept of climate as an ensemble distribution. The recently developed theory of non-autonomous dynamical systems was employed to mathematically analyse the alternative definitions of climate and to identify the main differences and similarities between the cases of constant and varying external conditions.

Ensemble distributions encounter more problems than distributions over time because the former may not relate to the time series of past observations, have difficulties conceptualising the past and the present climate and climate change and imply that the climate depends on our uncertainty. For all concepts of climate there is both a finite and an infinite version. It was argued that the finite versions are preferable because infinite distributions may be empirically void, the relevant mathematical limits may not exist and when there are two infinite distributions it is difficult to identify which of them is the climate. Hence the finite version of concept T3 (that is, the finite distribution over time under a certain regime of varying external conditions) is the most promising definition. Concept E2 is still useful, but one might say that instead of defining the climate, it just refers to a distribution which is useful for predictive purposes.

Finally, let us look at the definition of climate given by the IPCC report (Solomon et al., 2007, p. 942):

> Climate in a narrow sense is usually defined as the average weather, or more rigorously, as the statistical description in terms of the mean and variability of relevant quantities over a period of time ranging from months to thousands or millions of years. The classical period for averaging these variables is 30 years, as defined by the World Meteorological Organization. The relevant quantities are most often surface variables such as temperature, precipitation and wind. Climate in a wider sense is the state, including a statistical description, of the climate system.

ESDD
5, 683–719, 2014

“Climate in a narrow sense” is intended to refer to the distributions of a more limited set of climate variables and “climate in a wider sense” to the distributions of a more extended set of climate variables (cf. Sect. 2). Apart from this, the definition is vague and open to different interpretations. “Climate in a narrow sense” refers to a distribution over time (the most direct interpretation is that it refers to the finite version of concept T2, i.e. the finite distributions over time for the actual path of the external conditions). “Climate in a wider sense” could even be interpreted as referring to any of the concepts discussed in this paper.

This vagueness is probably intended to subsume the various different concepts of climate under one general definition. Still, as the paper has hopefully shown, to avoid conceptual confusion, it is important to choose a good and clear definition of climate.

Acknowledgements. Many thanks to Mickael Checkroun, Stefaan Willem Conradie, Gary Froyland, Casey Helgeson, Wendy Parker, Leonhard Smith and the people at the Grantham Institute on Climate Change and the Environment at the London School of Economics for valuable discussions and suggestions. Funding support for the research was provided by the Arts and Humanities Research Council and by the Economic and Social Research Council.

References

Allen, M.: Liability for climate change, Nature, 421, 891–892, 2010.

Berger, A.: Chaos and Chance, an Introduction to Stochastic Aspects of Dynamics, Walter de Gruyter, Berlin, 2001.

Bhattacharya, K., Ghil, M., and Vulis, I. L.: Internal variability of an energy-balance model with delayed albedo effects, J. Atmos. Sci., 39, 1747–1773, 1982.

Buzzi, J.: Exponential decay of correlations for random Lasota–Yorke maps, Commun. Math. Phys., 208, 25–54, 1999.

Checkroun, M. D., Simnnet, E., and Ghil, M.: Stochastic climate dynamics: random attractors and time-dependent invariant measures, Physica D, 240, 1685–1700, 2011.

Daron, J. D.: Examining the Decision-Relevance of Climate Model Information for the Insurance Industry, Ph.D. thesis, London School of Economics and Political Science, London, 2012.

ESDD
5, 683–719, 2014

Definitions of climate and climate change under varying external conditions

C. Werndl

Daron, J. D. and Stainforth, D.: On predicting climate under climate change, Environ. Res. Lett., 8, 1–8, 2013.
Dymnikov, V. P. and Gritsoun, A. S.: Climate model attractors: chaos, quasi-regularity and sensitivity to small perturbations of external forcing, Nonlin. Processes Geophys., 8, 201–209, doi:10.5194/npg-8-201-2001, 2001.
Eckmann, J.-P. and Ruelle, D.: Ergodic theory of chaos and strange attractors, Rev. Mod. Phys., 57, 617–656, 1985.
Gallavotti, G. and Cohen, E. G. D.: Dynamical ensembles in stationary states, J. Stat. Phys., 80, 931–970, 1995.
Goswami, B. N., Wu, G., and Yasunari, T.: The annual cycle, intraseasonal oscillations, and roadblock to seasonal predictability of the Asian summer monsoon, J. Climate, 19, 5078–5099, 2006.
Hulme, M., Dessai, S., Lorenzoni, I., and Nelson, D.: Unstable climates: exploring the statistical and social constructions of climate, Geoforum, 40, 197–206, 2009.
Jacobsen, M. V.: Absolutely continuous invariant measures for one-parameter families of one-dimensional maps, Commun. Math. Phys., 81, 39–88, 1981.
Jin, F.-F., Neelin, J. D., and Ghil, M.: El Niño on the devil's staircase: annual subharmonic steps to chaos, Science, 264, 70–72, 1994.
Kloeden, P. E. and Rasmussen, M.: Nonautonomous Dynamical Systems, Mathematical Surveys and Monographs, Vol. 176, American Mathematical Society, Providence, 2011.
Kurgansky, M. V., Dethloff, K., Pisnichenko, I. A., Gernandt, H., Chmielevski, F.-M., and Jansen, W.: Long term climate variability in a simple, nonlinear atmospheric model, J. Geophys. Res., 101, 4299–4314, 1996.
Lasota, A. and MacKey, M. C.: Probabilistic Properties of Deterministic Systems, Cambridge University Press, Cambridge, 1985.
Lorenz, E.: Deterministic nonperiodic flow, J. Atmos. Sci., 20, 130–141, 1963.
Lorenz, E.: The problem of deducing the climate from the governing equations, Tellus, 16, 1–11, 1964.
Lorenz, E.: Climatic determinism, Meteor. Mon., 8, 1–3, 1968.
Lorenz, E.: Climatic change as a mathematical problem, J. Appl. Meteorol., 9, 325–329, 1970.
Lorenz, E.: Nondeterministic theories of climatic change, Quaternary Res., 6, 495–506, 1976.
Lorenz, E.: Can chaos and intransitivity lead to interannual variability?, Tellus A, 42, 378–389, 1990.

ESDD
5, 683–719, 2014

Definitions of climate and climate change under varying external conditions

C. Werndl

Lorenz, E.: Climate is what you expect, prepared for publication by NCAR, unpublished, 1–33, 1995.
Lucarini, V.: Evidence of dispersion relations for the nonlinear response of the Lorenz 63 system, J. Stat. Phys., 134, 381–400, 1999.
Lucarini, V.: Stochastic perturbations to dynamical systems: a response theory approach, J. Stat. Phys., 146, 774–786, 2012.
Lucarini, V. and Sarno, S.: A statistical mechanical approach for the computation of the climatic response to general forcings, Nonlin. Processes Geophys., 18, 7–28, doi:10.5194/npg-18-7-2011, 2011.
Luzzatto, S., Melbourne, I., and Paccaut, F.: The Lorenz attractor is mixing, Commun. Math. Phys., 260, 393–401, 2005.
Lyubich, M.: Almost every quadratic map is either regular or stochastic, Ann. Math., 156, 1–78, 2002.
Mancho, A. M., Wiggins, S., Curbelo, J., and Mendoza, C.: Lagrangian descriptors: a method for revealing phase space structures of general time dependent dynamical systems, Commun. Nonlinear Sci., forthcoming, 2014.
McGuffie, K. and Henderson-Sellars, A.: A Climate Modelling Primer, John Wiley & Sons, Chichester, 2005.
Palmer, T.: A nonlinear dynamical perspective on climate prediction, J. Climate, 12, 575–591, 1999.
Parker, W.: Understanding pluralism in climate modeling, Found. Sci., 11, 349–368, 2006.
Petersen, K. E.: Ergodic Theory, Cambridge University Press, Cambridge, 1983.
Ruelle, D.: A measure associated with Axiom A attractors, Am. J. Math., 98, 619–654, 1976.
Ruelle, D.: A review of linear response theory for general differentiable dynamical systems, Nonlinearity, 22, 855–870, 2009.
Schneider, S. H. and Dickinson, R. E.: Climate modelling, Clim. Change, 45, 203–221, 2000.
Sempf, M., Dethloff, K., Handorf, D., Kurgansky, M. V.: Toward understanding the dynamical origin of atmospheric regime behavior in a baroclinic model, J. Atmos. Sci., 64, 887–904, 2007.
Smith, L.: What might we learn from climate forecasts, P. Natl. Acad. Sci. USA, 99, 2487–2492, 2002.
Solomon, S., Qin, D., Manning, M., Chen, Z., Marquis, M., Averyt, K. B., Tingor, M., and Miller, H. L.: Climate Change 2007: The Physical Science Basis, Contribution of Work-

ESDD
5, 683–719, 2014

Definitions of climate and climate change under varying external conditions

C. Werndl

ing Group I to the Fourth Assessment Report of the Intergovernmental Panel on Climate Change, IPCC Report, Cambridge University Press, Cambridge, 2007.

Stone, D. A. and Knutti, R.: Weather and climate, in: Modelling the Impact of Climate Change on Water Resources, edited by: Fung, F., Lopez, A., and New, M., John Wiley & Sons, Chichester, 4–33, 2010.

Stone, D. A., Allen, M. R., Stott, P. A., Pall, P., Min, S.-K., Nozawa, T., and Yukimoto, S.: The detection and attribution of human influence on climate, Annu. Rev. Env. Resour., 34, 1–16, 2009.

Tasaki, S., Gilbert, T., and Dorfman, J. R.: An analytical construction of the SRB measures for baker-type maps, Chaos, 8, 424–444, 1998.

Werndl, C.: What are the new implications of chaos for unpredictability?, Brit. J. Philos. Sci., 60, 195–220, 2009.

Werndl, C.: On defining climate and climate change, Brit. J. Philos. Sci., forthcoming, 2014.

Young, L.-S.: What are SRB measures, and which dynamical systems have them?, J. Stat. Phys., 108, 733–754, 2002.

ESDD

5, 683–719, 2014

Definitions of climate and climate change under varying external conditions

C. Werndl

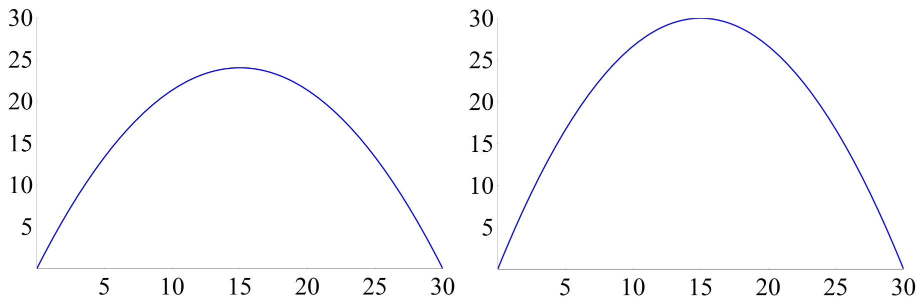

Figure 1. The evolution of the temperature when $\alpha_t = 3.2$ (left) and when $\alpha_t = 4$ (right).

ESDD
5, 683–719, 2014

Definitions of climate and climate change under varying external conditions

C. Werndl

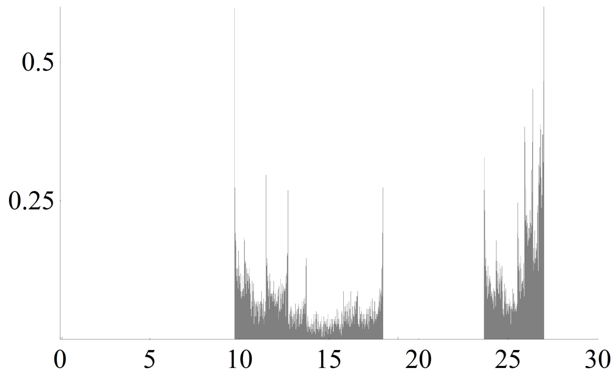

Figure 2. The temperature distribution over time for constant $c = 3.6$ from 1 July 1984 to 30 June 2014 and with initial value 25.25 for the simple climate model.

ESDD
5, 683–719, 2014

Definitions of climate and climate change under varying external conditions

C. Werndl

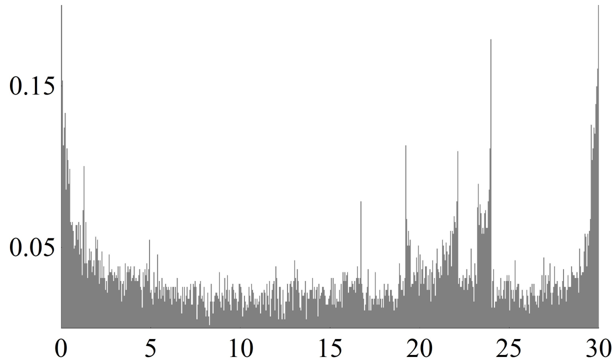

Figure 3. The actual temperature distribution over time from 1 July 1984 to 30 June 2014 and with initial value 25.25 for the simple climate model.

ESDD
5, 683–719, 2014

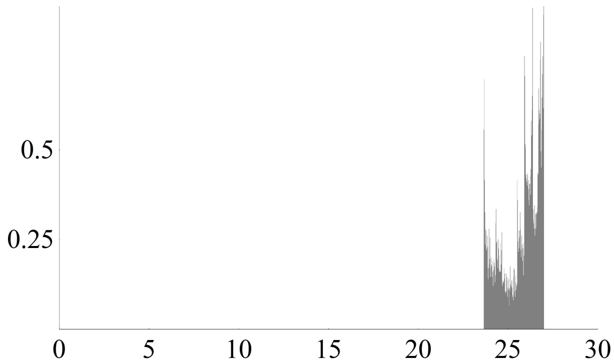

Figure 4. The ensemble distribution of the temperature on 1 July 2050 for the simple climate model with constant $c = 3.6$ conditional on the uncertainty about the initial temperature on 1 July 2014.

ESDD
5, 683–719, 2014

Definitions of climate and climate change under varying external conditions

C. Werndl

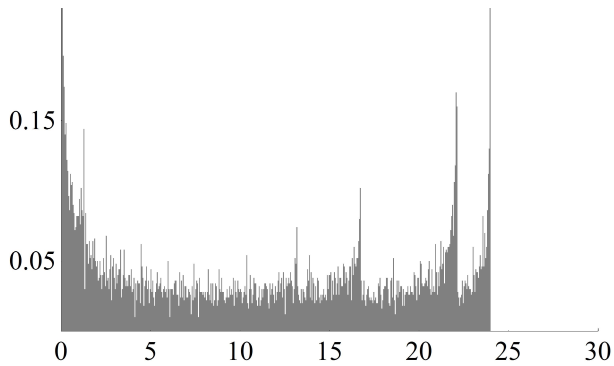

Figure 5. The actual ensemble distribution of the temperature on 1 July 2050 for the simple climate model conditional on the uncertainty about the initial temperature on 1 July 2014.

ESDD
5, 683–719, 2014