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Fractional governing equations of transient groundwater flow in confined aquifers with multi-fractional dimensions in fractional time

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Abstract. Using fractional calculus, a dimensionally consistent governing equation of transient, saturated groundwater flow in fractional time in a multi-fractional confined aquifer is developed. First, a dimensionally consistent continuity equation for transient saturated groundwater flow in fractional time and in a multi-fractional, multidimensional confined aquifer is developed. For the equation of water flux within a multi-fractional multidimensional confined aquifer, a dimensionally consistent equation is also developed. The governing equation of transient saturated groundwater flow in a multi-fractional, multidimensional confined aquifer in fractional time is then obtained by combining the fractional continuity and water flux equations. To illustrate the capability of the proposed governing equation of groundwater flow in a confined aquifer, a numerical application of the fractional governing equation to a confined aquifer groundwater flow problem was also performed.

1 Introduction

Previous laboratory and field studies (Levy and Berkowitz, 2003; Silliman and Simpson, 1987; Peaudecerf and Sauty, 1978; Sidle et al., 1998; Sudicky et al., 1983) demonstrated substantial deviations from Fickian behavior in transport in subsurface porous media. Various authors (Meerschaert et al., 1999, 2002, 2006; Benson et al., 2000a, b; Schumer et al., 2001, 2009; Baeumer et al., 2005; Baeumer and Meerschaert, 2007; Zhang et al., 2007, 2009; Zhang and Benson, 2008) have introduced the fractional advection–dispersion equation (fADE) as a model for transport in heterogeneous subsurface media as one approach to the modeling of the generally non-Fickian behavior of transport. As was demonstrated by the studies above, the heavy-tailed non-Fickian dispersion in subsurface media can be modeled well by a fractional spatial derivative, and the long particle waiting times in transport can be modeled well by means of a fractional time derivative within fADE. However, the abovementioned studies focused on the fractional differential equation modeling of solute transport in fractional time and space, and not on the modeling of the underlying subsurface flows that transport the solutes. Also, as shown by Kim et al. (2014), non-Fickian behavior in transport can also be obtained if the underlying flow field has a long memory in time, which can be described by a time-fractional governing equation of the specific flow field (Ercan and Kavvas, 2014, 2016). Kang et al. (2015) also showed that velocity correlation and distribution in fractured media may lead to non-Fickian transport and proposed a continuous-time random walk model (see Metzler and Klafter, 2000, for details of such models) that can account for velocity correlation and distribution.

Cloot and Botha (2006) argued that there are many fractured rock aquifers in which the groundwater flow does not fit conventional geometries (Black et al., 1986), and in such aquifers the conventional radial groundwater flow model underestimates the observed drawdown in early times and overestimates it at later times (Van Tonder et al., 2001). Based on this argument, which they supported with some radial flow field data, Cloot and Botha (2006) then formulated a fractional governing equation for radial groundwater flow in integer time but fractional space and provided some numerical applications of this model. In that formulation they also provided a formulation of the Darcy flux in radial fractional space. However, in addition to taking the time as an integer, they also considered a uniform homogeneous aquifer with a constant hydraulic conductivity. In the formulation of their radial groundwater flow model, they did not provide a derivation of the mass conservation equation for groundwater flow in fractional time and space. Also, they utilized the Riemann–Liouville form of the fractional derivative. Later, Atangana and his co-workers (Atangana, 2014; Atangana and Bildik, 2013; Atangana and Vermeulen, 2014) developed the fractional radial groundwater flow formulation of Cloot and Botha (2006) in terms of the Caputo derivative and claimed it yielded superior performance when compared to the Riemann–Liouville derivative formulation. The fundamental advantage of the Caputo derivative over the Riemann–Liouville derivative is that it can accommodate the real-life initial and boundary conditions, while the Riemann–Liouville derivative cannot (Podlubny, 1998). That is, the fractional differential equations with Caputo derivatives contain the physically interpretable integer-order derivatives at the initial times and at the upstream spatial boundaries, whereas the Riemann–Liouville derivatives do not (Podlubny, 1998). More recently, Atangana and Baleanu (2014) utilized a new definition of the fractional derivative, called the "conformable derivative" (Khalil et al., 2014), for the modeling of radial groundwater flow in fractional time but integer space. In all the studies above, the authors formulated their fractional governing equations instead of providing derivations of their groundwater flow equations from the basic conservation principles.

Wheatcraft and Meerschaert (2008) were the first to provide a comprehensive derivation of the continuity equation for groundwater flow. These authors have shown that since a first-order Taylor series approximation is used to represent the change in the mass flux through a control volume, the traditional continuity equation in an infinitesimal control volume is exact only when the change in flux in the control volume is linear. They also showed that, analogous to a first-order Taylor series, a fractional Taylor series is able to represent the nonlinear flux in a control volume by exactly only two terms. By replacing the integer-order Taylor series approximation for flux with the fractional-order Taylor series approximation, they derived a fractional form of the continuity equation for groundwater flow, removing the linearity or piecewise linearity restriction for the flux and the restriction that the control volume must be infinitesimal. In their development of the continuity equation, Wheatcraft and Meerschaert (2008) considered the porous medium in fractional space but the flow process in integer time. They also considered the fractional porous media space to have the same fractional power in all directions. Furthermore, their derivation is confined to only the mass conservation. It does not address the fractional water flux (motion) equation, nor the complete governing equation of groundwater flow.

Groundwater level fluctuations through time at certain locations exhibit long-range time correlation, which implies the need for the incorporation of time-fractional operation in the standard groundwater flow governing equations in order to accommodate the long-range time dependence (Li and Zhang, 2007; Rakhshandehroo and Amiri, 2012; Tu et al., 2017; Yu et al., 2016). Hence, in order to provide a general modeling structure, it is necessary to develop the governing equations of confined groundwater flow in fractional time as well as in fractional space. Also, different fractional powers should be considered in different spatial directions in order to accommodate the anisotropy of a confined aquifer medium.

In parallel to the conventional governing equations of groundwater flow processes (Bear, 1979; Freeze and Cherry, 1979), the corresponding time–space fractional governing equations of the confined groundwater flow must have certain characteristics (Kavvas et al., 2017): (a) from the outset, the form of the governing equation must be known completely. As such, it must be a prognostic equation. That is, in order to describe the evolution of the flow field in time and space it is solved from the initial conditions and boundary conditions. The governing equation is fixed throughout the simulation time and space for the simulation of the groundwater flow in question once its physical parameters, such as porosity, saturated hydraulic conductivity, etc., are estimated. (b) The fractional governing equations must be purely differential equations, containing only differential operators and no difference operators. (c) These equations must be dimensionally consistent. (d) As the orders of the fractional derivatives in the equations approach the corresponding integer powers, the fractional governing equations of confined groundwater flow with fractional powers must converge to the corresponding conventional governing equations with integer powers. The following development of the fractional governing equations of confined groundwater flow will be performed within the framework above.

2 Derivation of the continuity equation for transient groundwater flow in a multi-fractional confined aquifer in fractional time

Let $D_a^{k\beta} f(x)$ be a Caputo fractional derivative of the function $f(x)$, defined as (Li et al., 2009; Odibat and Shawagfeh, 2007; Podlubny, 1998; Usero, 2008)

$$D_a^{k\beta} f(x) = \frac{1}{\Gamma(m-k\beta)} \int_a^x \frac{f^m(\xi)}{(x-\xi)^{k\beta+1-m}} \mathrm{d}\xi, \quad m-1<\beta<m,\ m\epsilon N,\ x\geq a. \tag{1}$$

Specializing the integer $m=1$ reduces Eq. (1) to

$$D_a^{k\beta} f(x) = \frac{1}{\Gamma(1-k\beta)} \int_a^x \frac{f'(\xi)}{(x-\xi)^{k\beta}} \mathrm{d}\xi,\ 0<\beta<1,\ x\geq a, \tag{2}$$

then to β-order

$$D_a^{\beta} f(x) = \frac{1}{\Gamma(1-\beta)} \int_a^x \frac{f'(\xi)}{(x-\xi)^{\beta}} \mathrm{d}\xi, \; 0<\beta<1, \; x \geq a. \quad (3)$$

One can obtain a β_{x_i}-order approximation ($i=1, 2, 3$; $x_1=x$, $x_2=y$, $x_3=z$) to a function $f(\cdot)$ around a as

$$f(x_i) = f(a) + \frac{(x_i-a)^{\beta_{x_i}}}{\Gamma\left(\beta_{x_i}+1\right)} D_a^{\beta_{x_i}} f(x_i), \; 0<\beta_{x_i}<1; \quad i=1,2,3 x_1=x, \; x_2=y, \; x_3=z. \quad (4)$$

This result may be obtained by taking in the mean value representation of a function in terms of the fractional Caputo derivative (Odibat and Shawagfeh, 2007; Usero, 2008; Li et al., 2009) the upper limit value of the Caputo derivative at x_i ($i=1, 2, 3$; $x_1=x$, $x_2=y$, $x_3=$z) to have a distinct value for the β_{x_i}-order approximation above ($i=1, 2, 3$; $x_1=x$, $x_2=y$, $x_3=z$) of the function f around a. Based on this approximation, for the whole modeling domain in time and space, the governing equations become prognostic equations that shall be known from the outset of model simulation. The next issue is what to take for the value of a. If one expresses Eq. (4) with $a=x_i-\Delta x_i$, that is,

$$f(x_i) = f(x_i-\Delta x_i) + \frac{(\Delta x_i)^{\beta_{x_i}}}{\Gamma\left(\beta_{x_i}+1\right)} D_{x_i-\Delta x_i}^{\beta_{x_i}} f(x_i); \quad i=1,2,3; \; x_1=x, \; x_2=y, \; x_3=z, \quad (5)$$

then the question becomes what to take for the value of Δx_i in Eq. (5). In order to obtain fractional governing equations as purely differential equations, an analytical relationship between Δx_i and $(\Delta x_i)^{\beta}$ ($i=1, 2, 3$; $x_1=x$, $x_2=y$, $x_3=z$) that will be universally applicable throughout the modeling domain must be established. Such an analytical relationship is possible when the lower limit in the Caputo derivative above in Eq. (5) is taken as zero (that is, $\Delta x_i=x_i$) for $f(x_i)=x_i$. As will be shown below, it will be possible to develop purely differential forms (with no finite difference operators) for the fractional governing equations of confined groundwater flow by following the construct above.

The net mass flux through the control volume in Fig. 1, which also has a sink–source mass flux $q_v \Delta x \Delta y \Delta z$, can be formulated within the framework above as

$$\begin{aligned} &\left[\rho q_x(x,y,z;t) - \rho q_x(x-\Delta x,y,z;t)\right] \Delta y \Delta z \\ &+ \left[\rho q_y(x,y,z;t) - \rho q_y(x,y-\Delta y,z;t)\right] \Delta x \Delta z \\ &+ \left[\rho q_z(x,y,z;t) - \rho q_z(x,y,z-\Delta z;t)\right] \Delta x \Delta y \\ &+ \rho q_v \Delta x \Delta y \Delta z. \end{aligned} \quad (6)$$

Then combining Eq. (5) with Eq. (6) with $\Delta x_i=x_i (i=1, 2, 3; x_1=x, x_2=y, x_3=z)$ and expressing the resulting Caputo derivative $D_0^{\beta_{x_i}} f(x_i)$ (taking $\Delta x_i=x_i$ causes the lower limit

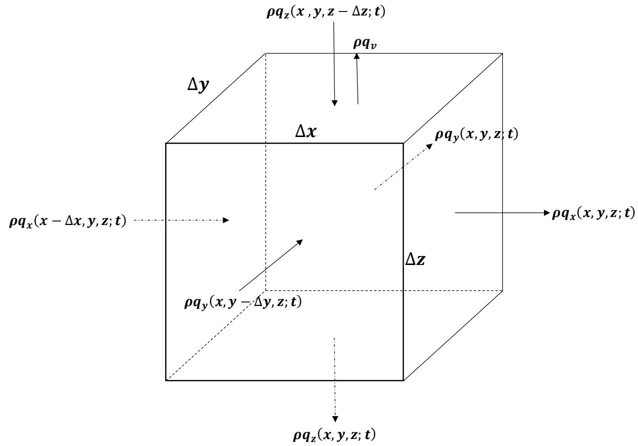

Figure 1. The control volume for the three-dimensional groundwater flow in confined aquifers.

in the Caputo derivative of Eq. 5 to become 0) by $\frac{\partial^{\beta_{x_i}} f(x_i)}{(\partial x_i)^{\beta_{x_i}}}$, ($i=1, 2, 3$; $x_1=x$, $x_2=y$, $x_3=z$) for convenience, yields the net mass flux through the control volume in Fig. 1 to the orders of $(\Delta x)^{\beta_x}$, $(\Delta y)^{\beta_y}$, and $(\Delta z)^{\beta_z}$ as

$$\begin{aligned} &\frac{1}{\Gamma(\beta_x+1)} \left(\frac{\partial}{\partial x}\right)^{\beta_x} (\rho q_x(x,y,z;t)) (\Delta x)^{\beta_x} \Delta y \Delta z \\ &+ \frac{1}{\Gamma\left(\beta_y+1\right)} \left(\frac{\partial}{\partial y}\right)^{\beta_y} \left(\rho q_y(x,y,z;t)\right) \Delta x (\Delta y)^{\beta_y} \Delta z \\ &+ \frac{1}{\Gamma(\beta_z+1)} \left(\frac{\partial}{\partial z}\right)^{\beta_z} (\rho q_z(x,y,z;t)) \Delta x \Delta y (\Delta z)^{\beta_z} \\ &+ \rho q_v \Delta x \Delta y \Delta z, \end{aligned} \quad (7)$$

where, due to the anisotropy in the hydraulic conductivities and in the subsequent flows in the porous media, different powers for fractional derivatives are considered in the three Cartesian directions in space.

From Eq. (5) it also follows with $f(x_i)=x_i$ that to the order of $(\Delta x_i)^{\beta_{x_i}}$, $i=1, 2, 3$,

$$\Delta x_i = \frac{(\Delta x_i)^{\beta_{x_i}}}{\Gamma\left(\beta_{x_i}+1\right)} \frac{\partial^{\beta_{x_i}} x_i}{(\partial x_i)^{\beta_{x_i}}}, \quad i=1,2,3; \; x_1=x, \; x_2=y, \; x_3=z. \quad (8)$$

Also for the Caputo derivative,

$$\frac{\partial^{\beta_{x_i}} x_i}{(\partial x_i)^{\beta_{x_i}}} = \frac{x_i^{1-\beta_{x_i}}}{\Gamma\left(2-\beta_{x_i}\right)}, \quad i=1,2,3; \; x_1=x, \; x_2=y, \; x_3=z. \quad (9)$$

Hence, introducing Eq. (9) into Eq. (8) results in β_{x_i}-order fractional increments in space in the ith direction, $i=1, 2, 3$,

$$(\Delta x_i)^{\beta_{x_i}} = \frac{\Gamma\left(\beta_{x_i}+1\right)\Gamma\left(2-\beta_{x_i}\right)}{x_i^{1-\beta_{x_i}}} \Delta x_i, \; x_1=x, \; x_2=y,$$

$$x_3 = z;\ \beta_{x_1} = \beta_x,\ \beta_{x_2} = \beta_y,\ \beta_{x_3} = \beta_z. \tag{10}$$

For the net mass outflow through the control volume in Fig. 1 (to the order of $(\Delta x_i)^{\beta_{x_i}}$, $i = 1, 2, 3$; $x_1 = x$, $x_2 = y$, $x_3 = z$), combining Eqs. (10) and (7) yields

$$\begin{aligned}&\frac{\Gamma(2-\beta_x)}{x^{1-\beta_x}}\left(\frac{\partial}{\partial x}\right)^{\beta_x}(\rho q_x(\overline{x};t))\Delta x\Delta y\Delta z\\ &+\frac{\Gamma\left(2-\beta_y\right)}{y^{1-\beta_y}}\left(\frac{\partial}{\partial y}\right)^{\beta_y}\left(\rho q_y(\overline{x};t)\right)\Delta y\Delta x\Delta z\\ &+\frac{\Gamma(2-\beta_z)}{z^{1-\beta_z}}\left(\frac{\partial}{\partial z}\right)^{\beta_z}(\rho q_z(\overline{x};t))\Delta z\Delta x\Delta y\\ &+\rho q_v\Delta x\Delta y\Delta z,\ \overline{x}=(x,y,z).\end{aligned} \tag{11}$$

Denoting the porosity, which is the water volume per volume of the control volume in Fig. 1 under saturated conditions, using n, the change in mass within the control volume in Fig. 1 per time increment Δt may be expressed as (Freeze and Cherry, 1979)

$$(\rho n|_t - \rho n|_{t-\Delta t})/\Delta t. \tag{12}$$

Meanwhile, the specific storage S_s of a saturated aquifer may be defined as the volume of water that is released from a unit volume of the aquifer under a unit decline in the hydraulic head h (Freeze and Cherry, 1979). Under this definition the change in mass in the control volume of Fig. 1 per time increment Δt may be expressed as (Freeze and Cherry, 1979)

$$\begin{aligned}\frac{(\rho n|_t-\rho n|_{t-\Delta t})}{\Delta t}&=\frac{\rho S_s(h|_t-h|_{t-\Delta t})}{\Delta t}\Delta x\Delta y\Delta z\\ &=\rho S_s\frac{\Delta h}{\Delta t}\Delta x\Delta y\Delta z.\end{aligned} \tag{13}$$

Expressing the relationship (Eq. 10) to α-order fractional increments in time,

$$(\Delta t)^\alpha = \frac{\Gamma(\alpha+1)\Gamma(2-\alpha)}{t^{1-\alpha}}\Delta t. \tag{14}$$

Meanwhile, using the approximation (Eq. 5) in the time dimension to the order of $(\Delta t)^\alpha$, for any function g of time,

$$g(t) - g(t-\Delta t) = \frac{(\Delta t)^\alpha}{\Gamma(\alpha+1)}\left(\frac{\partial}{\partial t}\right)^\alpha g(t). \tag{15}$$

Introducing Eq. (15) into the right-hand side (RHS) of Eq. (13) yields to the order of $(\Delta t)^\alpha$,

$$\rho S_s\frac{1}{\Delta t}\frac{(\Delta t)^\alpha}{\Gamma(\alpha+1)}\left(\frac{\partial}{\partial t}\right)^\alpha (h)\Delta x\Delta y\Delta z. \tag{16}$$

Then introducing Eq. (14) into the expression (Eq. 16) yields

$$\rho S_s\frac{\Gamma(2-\alpha)}{t^{1-\alpha}}\left(\frac{\partial}{\partial t}\right)^\alpha (h)\Delta x\Delta y\Delta z \tag{17}$$

as the time rate of change of mass in the control volume of size $\Delta x\Delta y\Delta z$.

Since the net flux through the control volume is inversely related to the time rate of change of mass within the control volume of Fig. 1, one may combine Eqs. (11) and (17) to obtain

$$\begin{aligned}\rho S_s\frac{\Gamma(2-\alpha)}{t^{1-\alpha}}\left(\frac{\partial}{\partial t}\right)^\alpha(h)=-\Bigg[&\frac{\Gamma(2-\beta_x)}{x^{1-\beta_x}}\left(\frac{\partial}{\partial x}\right)^{\beta_x}\\ &(\rho(\overline{x};t)q_x(\overline{x};t))+\frac{\Gamma\left(2-\beta_y\right)}{y^{1-\beta_y}}\left(\frac{\partial}{\partial y}\right)^{\beta_y}\\ &\left(\rho(\overline{x};t)q_y(\overline{x};t)\right)+\frac{\Gamma(2-\beta_z)}{z^{1-\beta_z}}\left(\frac{\partial}{\partial z}\right)^{\beta_z}\\ &(\rho(\overline{x};t)q_z(\overline{x};t))+\rho q_v\Bigg].\end{aligned} \tag{18}$$

In the conventional case with the integer derivatives (Freeze and Cherry, 1979),

$$\rho\frac{\partial q_{x_i}}{\partial x_i} \gg q_{x_i}\frac{\partial\rho}{\partial x_i},\ i=1,2,3;\ x_1=x,\ x_2=y,\ x_3=z. \tag{19}$$

Hence, it is also expected that

$$\begin{aligned}&\rho\frac{\partial^{\beta_i}q_{x_i}}{(\partial x_i)^{\beta_i}} \gg q_{x_i}\frac{\partial^{\beta_i}\rho}{(\partial x_i)^{\beta_i}},\ i=1,2,3;\ x_1=x,\ x_2=y,\\ &x_3=z;\ \beta_1=\beta_x,\ \beta_2=\beta_y,\ \beta_3=\beta_z.\end{aligned} \tag{20}$$

Combining the inequality (Eq. 20) with Eq. (18) yields

$$\begin{aligned}S_s\frac{\partial^\alpha h}{(\partial t)^\alpha}=&-\frac{\Gamma(2-\beta_x)}{\Gamma(2-\alpha)}\frac{t^{1-\alpha}}{x^{1-\beta_x}}\left(\frac{\partial}{\partial x}\right)^{\beta_x}(q_x(\overline{x};t))\\ &-\frac{\Gamma\left(2-\beta_y\right)}{\Gamma(2-\alpha)}\frac{t^{1-\alpha}}{y^{1-\beta_y}}\left(\frac{\partial}{\partial y}\right)^{\beta_y}\left(q_y(\overline{x};t)\right)\\ &-\frac{\Gamma(2-\beta_z)}{\Gamma(2-\alpha)}\frac{t^{1-\alpha}}{z^{1-\beta_z}}\left(\frac{\partial}{\partial z}\right)^{\beta_z}(q_z(\overline{x};t))\\ &-q_v\frac{t^{1-\alpha}}{\Gamma(2-\alpha)}\ 0<\alpha,\ \beta_x,\ \beta_y,\ \beta_z<1,\\ &\overline{x}=(x_1,\ x_2,\ x_3)\end{aligned} \tag{21}$$

as the time–space fractional continuity equation of transient saturated groundwater flow in an anisotropic confined aquifer with fractional dimensions and in fractional time.

Performing a dimensional analysis of Eq. (21), one obtains

$$\begin{aligned}\frac{1}{T^\alpha}&=\frac{1}{L}\cdot\frac{L}{T^\alpha}=\frac{T^{1-\alpha}}{L^{1-\beta_x}}\frac{1}{L^{\beta_x}}\frac{L}{T}=\frac{T^{1-\alpha}}{L^{1-\beta_y}}\frac{1}{L^{\beta_y}}\frac{L}{T}\\ &=\frac{T^{1-\alpha}}{L^{1-\beta_z}}\frac{1}{L^{\beta_z}}\frac{L}{T}=\frac{1}{T^\alpha},\end{aligned} \tag{22}$$

where L denotes length and T denotes time. Hence, the left-hand side (LHS) and RHS of the continuity Eq. (21) for transient groundwater flow in multi-fractional space and fractional time are shown to be consistent by means of Eq. (22).

It was shown by Podlubny (1998) that for $n-1<\alpha$, $\beta_i<n$, where n is any positive integer, as α and $\beta_i \to n$, the Caputo fractional derivative of a function $f(y)$ to order α or $\beta_i (i=1,\ 2,\ 3;\ \beta_1=\beta_x,\ \beta_2=\beta_y,\ \beta_3=\beta_z)$ becomes the conventional nth derivative of the function $f(y)$. Specializing the Podlubny (1998) result to $n=1$, for α and $\beta_i \to 1$ $(i=1,2,3;\ \beta_1=\beta_x,\ \beta_2=\beta_y,\ \beta_3=\beta_z)$, reduces the continuity Eq. (21) to the conventional continuity equation for transient groundwater flow in a confined aquifer:

$$S_s\frac{\partial h}{\partial t}=-\frac{\partial}{\partial x}\left(q_x(\overline{x};t)\right)-\frac{\partial}{\partial y}\left(q_y(\overline{x};t)\right)-\frac{\partial}{\partial z}\left(q_z(\overline{x};t)\right)-q_v. \quad (23)$$

3 An equation for specific discharge (motion equation) in fractional multidimensional confined aquifers

A governing equation for water flux (specific discharge) q_{x_i}, $(i=1,\ 2,\ 3;\ x_1=x,\ x_2=y,\ x_3=z)$ in a saturated or unsaturated porous medium with fractional dimensions was recently developed (Kavvas et al., 2017). For the case of transient saturated groundwater flow in an anisotropic confined aquifer with multi-fractional dimensions, that equation for the specific discharge takes the form

$$q_i(\overline{x},t)=-K_{s,x_i}(\overline{x})\frac{\Gamma(2-\beta_i)}{x_i^{1-\beta_i}}\frac{\partial^{\beta_i}h}{(\partial x_i)^{\beta_i}},\ i=1,2,3;$$
$$x_1=x,\ x_2=y,\ x_3=z, \quad (24)$$

where $K_{s,x_i}(\overline{x})$ denotes the saturated hydraulic conductivity in the ith spatial direction $(i=1,2,3;x_1=x,x_2=y,x_3=z)$. Due to the groundwater flow being in the direction of decreasing hydraulic head, the RHS of Eq. (24) takes a negative sign.

A dimensional analysis on Eq. (24) yields L/T for the units of both the LHS and the RHS of the equation, establishing its dimensional consistency.

Applying the abovementioned result of Podlubny (1998) for the convergence of a fractional derivative to a corresponding integer derivative, for $\beta_i \to 1$ $(i=1,2,3;\ \beta_1=\beta_x,\ \beta_2=\beta_y,\ \beta_3=\beta_z)$, reduces the fractional specific discharge (Eq. 24) for groundwater flow to the conventional Darcy equation for groundwater specific discharge:

$$q_i(\overline{x},t)=-K_{s,x_i}(\overline{x})\frac{\partial h(\overline{x},t)}{\partial x_i},\ i=1,2,3;\ x_1=x,$$
$$x_2=y,\ x_3=z \quad (25)$$

for the case of integer spatial dimensions. As such, the fractional specific discharge (Eq. 24) for confined groundwater flow in fractional spatial dimensions is consistent with the conventional Darcy equation for the integer spatial dimensions.

4 The complete equation for transient confined groundwater flow in multi-fractional space and fractional time

One can combine the specific discharge Eq. (24) for groundwater flow (the motion equation) in a fractional confined aquifer with the time–space fractional continuity Eq. (21) of groundwater flow in fractional time and space in confined aquifers to obtain

$$S_s\frac{\partial^\alpha h}{(\partial t)^\alpha}=\frac{\Gamma(2-\beta_x)}{x^{1-\beta_x}}\left(\frac{\partial}{\partial x}\right)^{\beta_x}\left(K_{s,x}(\overline{x})\frac{t^{1-\alpha}}{x^{1-\beta_x}}\frac{\Gamma(2-\beta_x)}{\Gamma(2-\alpha)}\frac{\partial^{\beta_x}h}{(\partial x)^{\beta_x}}\right)+\frac{\Gamma(2-\beta_y)}{y^{1-\beta_y}}\left(\frac{\partial}{\partial y}\right)^{\beta_y}\left(K_{s,y}(\overline{x})\frac{t^{1-\alpha}}{y^{1-\beta_y}}\frac{\Gamma(2-\beta_y)}{\Gamma(2-\alpha)}\frac{\partial^{\beta_y}h}{(\partial y)^{\beta_y}}\right)+\frac{\Gamma(2-\beta_z)}{z^{1-\beta_z}}\left(\frac{\partial}{\partial z}\right)^{\beta_z}\left(K_{s,z}(\overline{x})\frac{t^{1-\alpha}}{z^{1-\beta_z}}\frac{\Gamma(2-\beta_z)}{\Gamma(2-\alpha)}\frac{\partial^{\beta_z}h}{(\partial z)^{\beta_z}}\right)-q_v\frac{t^{1-\alpha}}{\Gamma(2-\alpha)};$$
$$0<\alpha,\ \beta_x,\ \beta_y,\ \beta_z<1;\ \overline{x}=(x_1,\ x_2,\ x_3) \quad (26)$$

as the time–space fractional governing equation of transient saturated groundwater flow in a confined anisotropic aquifer with multi-fractional dimensions and in fractional time. In Eq. (26), q_v may be taken as the pumping rate or recharge rate.

Performing a dimensional analysis on the governing fractional Eq. (26) for confined groundwater flow results in

$$\frac{1}{T^\alpha}=\frac{1}{L^{1-\beta_x}}\frac{1}{L^{\beta_x}}\frac{L}{T}\frac{T^{1-\alpha}}{L^{1-\beta_x}}\frac{L}{L^{\beta_x}}=\frac{1}{L^{1-\beta_y}}\frac{1}{L^{\beta_y}}\frac{L}{T}\frac{T^{1-\alpha}}{L^{1-\beta_y}}\frac{L}{L^{\beta_y}}=\frac{1}{L^{1-\beta_z}}\frac{1}{L^{\beta_z}}\frac{L}{T}\frac{T^{1-\alpha}}{L^{1-\beta_z}}\frac{L}{L^{\beta_z}}=\frac{1}{T^\alpha}, \quad (27)$$

which shows that both the RHS and the LHS of the equation have the unit $\frac{1}{T^\alpha}$, which verifies its dimensional consistency.

Applying the abovementioned result of Podlubny (1998) for the convergence of a fractional derivative to a corresponding integer derivative, for α and $\beta_i \to 1$ $(i=1,\ 2,\ 3;$ $\beta_1=\beta_x,\ \beta_2=\beta_y,\ \beta_3=\beta_z)$, the governing Eq. (26) for confined groundwater flow in fractional time and space takes the form

$$S_s\frac{\partial h(\overline{x};t)}{\partial t}=\frac{\partial}{\partial x}\left(K_{s,x}(\overline{x})\frac{\partial h(\overline{x};t)}{\partial x}\right)+\frac{\partial}{\partial y}\left(K_{s,y}(\overline{x})\frac{\partial h(\overline{x};t)}{\partial y}\right)+\frac{\partial}{\partial z}\left(K_{s,z}(\overline{x})\frac{\partial h(\overline{x};t)}{\partial z}\right)-q_v,\ \overline{x}=(x_1,\ x_2,\ x_3), \quad (28)$$

which is the conventional governing equation for transient saturated groundwater flow in an anisotropic confined aquifer (Freeze and Cherry, 1979). As such, the time–space fractional governing Eq. (26) of transient groundwater flow in a confined anisotropic aquifer with multi-fractional dimensions in fractional time is consistent with the conventional governing equation for transient groundwater flow in an anisotropic confined aquifer with integer derivatives.

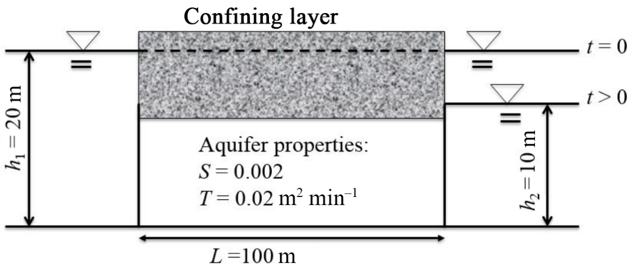

Figure 2. The reservoir example modified based on Wang and Anderson (1995).

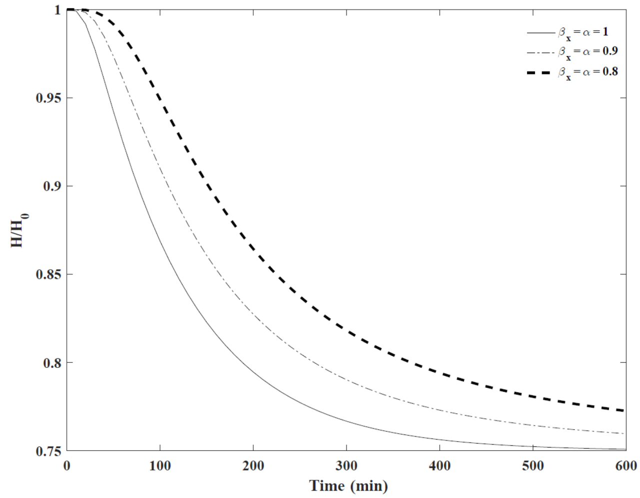

Figure 3. Nondimensional groundwater hydraulic heads through time at $x = L/2$, when fractional space and time derivatives are $\beta_x = \alpha = 0.8$, 0.9, 1.0, where L is the length of the aquifer and β_x and α are the fractional orders in space and time, respectively.

5 Physical meaning of fractional time derivative in the fractional governing equations of confined transient groundwater flow

Let us consider the Caputo fractional time derivative of the function $f(t)$,

$$\frac{\partial^\alpha f}{(\partial t)^\alpha} = D_0^\alpha f(t), \tag{29}$$

defined by

$$D_0^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(s)}{(t-s)^\alpha} \mathrm{d}s, \; 0 < \alpha < 1, \; t \geq 0. \tag{30}$$

As such, each local integer derivative $f'(s)$ at each time position s $(0 \leq s \leq t)$ in the time interval $(0, t)$ contributes with the weight $(t-s)^{-\alpha}$ to the Caputo fractional derivative of $f(t)$ during the time interval $(0, t)$. Hence, the Caputo derivative is a nonlocal quantity, pertaining to a time interval, vs. the conventional derivative of $f(t)$, $f'(t)$, which is defined for the particular time location t. Within this framework, the effect of the initial condition at the initial time location 0 is still accounted for at any time t $(0 \leq t \leq T)$ during the whole simulation period $(0, T)$ by means of the fractional time derivative that appears in the governing Eq. (26) above of confined transient groundwater flow in fractional time. It also follows from Eq. (30) that this memory effect is modulated by the value of the fractional power α. As shown by Podlubny (1998), as $\alpha \to 1$, the Caputo fractional time derivative of $f(t)$, as given by Eq. (30), converges to the local time derivative $f'(t)$ at t.

6 A numerical application of the developed fractional governing equation of confined groundwater flow

To illustrate the capability of the proposed governing equation of groundwater flow in a confined aquifer, a numerical application of the fractional governing equation to the physical setting of an example from Wang and Anderson (1995) is provided as shown in Fig. 2. In this example, groundwater flow in a confined aquifer is simplified to be one-dimensional. The length of the confined aquifer is 100 m. The hydraulic transmissivity (T) of the aquifer is 0.02 $\mathrm{m^2\,min^{-1}}$ and the specific storage (S) of the aquifer is 0.002. The groundwater hydraulic head is initially uniform at 20 m. The water level downstream suddenly drops to 10 m and stays at 10 m. The groundwater level upstream is set to be 20 m throughout the simulation duration. The total simulation time is 600 min.

Nondimensional groundwater hydraulic heads (H/H_0, where H_0 is the initial groundwater hydraulic head) at $x = 50$ m through time in the aquifer are shown in Fig. 3, in which fractional derivatives in space and time are taken as $\beta_x = \alpha = 0.8$, 0.9, 1.0. As one can see from Fig. 3, compared to the curve of hydraulic head recession in time that corresponds to $\beta_x = \alpha = 1.0$ (the conventional integer derivative case), the hydraulic head recession in time gets slower with the decrease in $\beta_x = \alpha$ from 1. The groundwater hydraulic heads in Fig. 3 clearly show heavier tails as fractional derivative orders in space and time decrease from 1. Additionally, the smaller the fractional orders are, the heavier the tails become with the increase in time. The modeling results may indicate nonlocal effects in groundwater flow and help explain the long-range dependence characteristics in some groundwater level fluctuation datasets (Tu et al., 2017). The results may also shed light on the non-Fickian transport phenomena in groundwater flow.

7 Discussion on the developed fractional governing equations in the context of broader geosciences

The conventional governing equations of porous media flows in geosciences in various environments are all local-scale equations in which only the interactions among nearest neighbors in time and space are described. All of these governing equations are differential equations where the powers of the derivative terms that appear in these equations take integer values. In the case that a porous media flow field shows interactions among time–space locations that are separated by substantial distances in time or space, the local-scale conventional governing flow equations for such media, because they are based on local interactions, may not be able to describe such long-distance interactions adequately. A more efficient approach for modeling such long-distance interactions in time and space may be the use of fractional governing equations of porous media flows. Such fractional governing equations, as those developed in this study, utilize time–space derivatives with fractional powers. As already shown in Sect. 5 above, the fractional Caputo time derivative is nonlocal, and, as such, can accommodate the effect of the initial conditions on the groundwater flow process for times that are substantially later than the initial time. Similarly, the fractional Caputo space derivatives in the governing Eqs. (21), (24), and (26) of this study are also nonlocal derivatives. To observe this, consider the Caputo fractional space derivative $D_0^{\beta} f(x_i)$:

$$D_0^{\beta} f(x_i) = \frac{1}{\Gamma(1-\beta)} \int_0^{x_i} \frac{f'(\xi)}{(x_i-\xi)^{\beta}} \mathrm{d}\xi. \tag{31}$$

Hence, each local integer derivative $f'(\xi)$ at each spatial location ξ in the spatial interval $(0, x_i)$ will contribute to the Caputo fractional derivative of the interval $(0, x_i)$ with the weight $(x_i-\xi)^{-\beta}$. As such, for groundwater flow in any i direction, the effect of a boundary condition that is placed at boundary location 0 in the i direction will be accounted for at any distance x_i from the boundary location 0 by means of the fractional space derivative that appears in the fractional governing equations above for the ith direction. It follows from Eq. (31) that this effect will be modulated by the value of the fractional derivative power β due to the weight $(x_i-\xi)^{-\beta}$.

As shown in the previous sections, the fractional governing equations converge to their conventional integer counterparts as the fractional derivative powers take integer values. Consequently, the conventional governing equations of porous media flows may be considered as special cases of the corresponding fractional governing equations, corresponding to the integer values of the derivative powers. While the fractional powers of the derivatives in the governing Eq. (26) may take any fractional value within the interval (0, 1), the integer powers of the derivatives in the conventional governing Eq. (28) are restricted to the value of unity. Within this context, the fractional governing equations of porous media flows may be thought of as the generalizations of the conventional governing equations of porous media flows with integer powers.

From the information above, it follows that the fractional governing equations developed in this study are nonlocal. Accordingly, they can account for the influence of the initial and boundary conditions on the flow process more effectively than the corresponding local-scale integer-order conventional governing equations since the conventional governing equations consider the effect of initial and boundary conditions on the flow processes within shorter time–space ranges.

From Eq. (28) it may be noted that the saturated hydraulic conductivity plays the role of a diffusion coefficient in the conventional governing equation of transient groundwater flow in an anisotropic confined aquifer in integer time and space. For discussion purposes, let us rewrite Eq. (26) for the governing equation of transient saturated groundwater flow in an anisotropic confined aquifer in fractional time and space:

$$\begin{aligned} S_\mathrm{s}\frac{\partial^{\alpha} h}{(\partial t)^{\alpha}} = {} & \frac{\Gamma(2-\beta_x)}{x^{1-\beta_x}}\left(\frac{\partial}{\partial x}\right)^{\beta_x} \\ & \left(K_{\mathrm{s},x}(\overline{x})\frac{t^{1-\alpha}}{x^{1-\beta_x}}\frac{\Gamma(2-\beta_x)}{\Gamma(2-\alpha)}\frac{\partial^{\beta_x} h}{(\partial x)^{\beta_x}}\right) + \frac{\Gamma(2-\beta_y)}{y^{1-\beta_y}}\left(\frac{\partial}{\partial y}\right)^{\beta_y} \\ & \left(K_{\mathrm{s},y}(\overline{x})\frac{t^{1-\alpha}}{y^{1-\beta_y}}\frac{\Gamma(2-\beta_y)}{\Gamma(2-\alpha)}\frac{\partial^{\beta_y} h}{(\partial y)^{\beta_y}}\right) + \frac{\Gamma(2-\beta_z)}{z^{1-\beta_z}}\left(\frac{\partial}{\partial z}\right)^{\beta_z} \\ & \left(K_{\mathrm{s},z}(\overline{x})\frac{t^{1-\alpha}}{z^{1-\beta_z}}\frac{\Gamma(2-\beta_z)}{\Gamma(2-\alpha)}\frac{\partial^{\beta_z} h}{(\partial z)^{\beta_z}}\right) - q_v\frac{t^{1-\alpha}}{\Gamma(2-\alpha)}; \\ & 0<\alpha,\ \beta_x,\ \beta_y,\ \beta_z<1;\ \overline{x}=(x_1,\ x_2,\ x_3). \end{aligned} \tag{32}$$

In this governing equation of transient confined groundwater flow in fractional time and space, the saturated hydraulic conductivities are augmented by fractional powers of time, $t^{1-\alpha}$, and of space, $x_i^{1-\beta_{x_i}}$, $i=1, 2, 3$, in terms of the ratios of fractional time to fractional space, $\frac{t^{1-\alpha}}{x_i^{1-\beta_{x_i}}}$, $i=1, 2, 3$, in multiple dimensions. As such the confined groundwater diffusion in fractional time and space is modulated by the ratios of fractional time to fractional space above. Accordingly, since the diffusion coefficient scales with a fractional power of time and a fractional power of space, the process represented by Eq. (32) may be thought to be non-Fickian. One can also see from Fig. 3 on the numerical application of the fractional confined groundwater flow equation to a simple one-dimensional case, as the fractional powers of the derivatives in space and time in the governing equation decrease from unity, the recession rate of the nondimensional hydraulic heads from the initial condition also becomes slower with respect to the case of the conventional governing equation with integer derivative powers. Therefore, the speed of the response of the groundwater system to the external forcings to the system (pumping rates, recharge rates, etc.) can be modulated in the fractional governing Eq. (26) of confined aquifer groundwater flow by means of the values that the fractional derivative power α takes, slowing down with the decrease in the values of α.

Kavvas et al. (2014) argued, and Kim et al. (2014) have shown by numerical simulations, that non-Fickian behavior in solute transport can also be obtained if the underlying flow field has a long memory, which can be described by a fractional governing equation of the specific flow field. Ercan and Kavvas (2014, 2016) have shown by numerical simulations that it is possible to obtain long waves in time and in space by means of the fractional governing equations of unsteady open channel flow.

8 Conclusion

In this study, a dimensionally consistent continuity equation for transient saturated groundwater flow in multi-fractional, multidimensional confined aquifers in fractional time was developed. It was then shown that as the fractional powers of time and space derivatives approach unity, the time–space fractional continuity equation approaches the conventional continuity equation for transient groundwater flow in a confined aquifer. For the motion equation of confined saturated groundwater flow, or the equation of water flux within a multi-fractional multidimensional confined aquifer, a dimensionally consistent equation was also developed. It was shown that as the fractional powers of the spatial derivatives approach unity, the fractional water flux equation approaches the conventional Darcy equation for groundwater specific discharge.

The governing equation of transient saturated groundwater flow in multi-fractional, multidimensional confined aquifers and in fractional time was then obtained by combining the fractional continuity and water flux equations. It was then shown that as the fractional powers of time and space derivatives approach unity, the time–space fractional governing equation of transient saturated confined groundwater flow approaches the conventional governing equation with integer derivatives for transient saturated groundwater flow in an anisotropic confined aquifer.

To illustrate the capability of the proposed governing equation of groundwater flow in a confined aquifer, a numerical application of the fractional governing equation to a confined aquifer groundwater flow problem was also performed. The modeling results indicate that the proposed governing equations may help explain the nonlocal effects in groundwater flow and may further help illustrate the associated non-Fickian transport in groundwater flow.

Data availability. The data used in this article can be accessed by contacting the corresponding author.

Competing interests. The authors declare that they have no conflict of interest.

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