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Supplement of

Topology of sustainable management of dynamical systems with desirable states: from defining planetary boundaries to safe operating spaces in the Earth system

J. Heitzig et al.

Correspondence to: J. Heitzig (heitzig@pik-potsdam.de)

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Supplementary Information

Supplement 1: Competing plant types model design

Although it is known that many plants modify the soil in ways that benefit their own growth, e.g. via influencing microbial communities and biogeochemical cycling (e.g., Kourtev et al. (2002); Read et al. (2003)) and empirical evidence exists that this has effects on interspecies plant competition (e.g., Poon (2011)), we know of no formal model that would allow to study the resulting feedbacks between two plants and is simple enough for the purpose of illustrating our theory in an adequate amount of space. The best existing candidate models seem to be the four-dimensional model of a two-species plant-soil-feedback by Bever (2003) (see also Kulmatiski et al. (2011)) and the spatially resolved model of an invading plant by Levine et al. (2006), which however does not model other species explicitly. For this reason, we chose to design a conceptual model of two fictitious plant types each of which grows according to the well-established logistic growth dynamics leading to an initially exponential growth that is dampened by intraspecies competition. In order to keep the state space dimension at only two dimensions so that state space diagrams can be plotted, we refrained from modelling the soil characteristics via dynamic variables as in the other models, and instead represented the soil modification effect by simply assuming that the two species' undampened growth rates are proportional to some carrying capacities K_1, K_2 that the current soil composition implies for the two species, and that K_1, K_2 depend directly on the existing two populations x_1, x_2 in some simple way. In order to study the effect of soil modification alone, we did not include other interspecies interactions such as direct interspecies competition for resources. Levine et al. (2006) also assume dampened growth with a basic rate that depends on the existing population, but they only focus on a single species and assume a fixed carrying capacity, which we find somewhat implausible in view of the empirical evidence presented in Poon (2011). Because we wanted to produce a conceptual model that illustrates the topological landscape in a multistable system, we needed to make sure the actual functional form we chose for K_1, K_2 produces a multistable system. This was achieved by assuming that the effect of the two populations x_1, x_2 on the two carrying capacities K_1, K_2 is nonlinear in the sense that the marginal soil improvement by plants of the same species is declining with higher populations while the marginal effect of plants of the other species is increasing with their population. We are not claiming that this is so in real-world plant-soil-feedback systems, but believe that the alternative assumption of a linear relationship seems unlikely. We then chose a very simple formula for K_1, K_2 that has these properties:

$$K_1(x_{1,2}) = \sqrt{x}_1(1 - x_2) \leqslant 1,$$
$$K_2(x_{1,2}) = \sqrt{x}_2(1 - x_1) \leqslant 1.$$

Supplement 2: Complete main cascade example

We include this synthetic example (without figure) to show that all of the regions from the main cascade and the manageable partition may be nonempty at once. In order to produce eddies, it needs to be at least two-dimensional. For simplicity, our example has a circularly symmetric default dynamics in 2D polar coordinates r,ϕ:

$$\dot{r} = f(r) = -\frac{r(r-2)(r-3)(r-5)(r-6)(r-8)(r-11)}{(9+r)^3},$$

$$\dot{\phi} = g(r) = r(r-5.5)(r-8)(r-8.5)(r-11)/100.$$

It has a stable fixed point at $r=0$, stable limit cycles at $r \in \{3,6,11\}$, unstable ones at $r \in \{2,5,8\}$, and changes in rotational direction at $r \in \{5.5, 8.5\}$ (between limit cycles) and on the stable limit cycles at $r \in \{8,11\}$.

We assume the management options are so that the admissible trajectories are those with $\dot{r} \in [f(r)-1/5, f(r)+1/5]$ and $\dot{\phi} = g(r)$, i.e., one can row only radially, with a relative speed of at most $1/5$ and arbitrarily large acceleration. For r in the intervals $R_1 \approx [.01,1.8]$, $R_2 \approx [3.65,4.05]$, $R_3 \approx [6.7,7.7]$, and $R_4 \approx [11.05,\infty)$, we have $f(r) < -1/5$ so that no stopping or rowing "outwards" is possible in the corresponding rings, while rowing "inwards" is always possible. If we choose the sunny region to be the (not circularly symmetric) half-plane $X^+ = \{y = r\sin\phi > 1\}$, then the upstream U is the interior of the region outside R_3, with approx. $r > 7.7$; the downstream D is the half-open ring between the outer bounds of R_2 and R_3, with approx. $r \in (4.05, 7.7]$; the unique trench is slightly larger than the disc $r \leqslant 1$; the unique abyss is approx. the ring with $r \in (1,1.8)$ including R_1; and the unique eddy equals approx. the ring with $r \in [1.8, 4.05]$ including R_2.

Supplement 3: Relationship to viability theory

The vast mathematical literature on viability theory (VP), summarized in (Aubin, 2009; Aubin et al., 2011), also treats the question of which regions of state space can be reached from which others when a system's dynamics has some additional degrees of freedom that may represent unknown internal components such as human behaviour, or unknown external drivers, or options for management or control.

Its fundamental concepts of *viable domain, viability kernel,* and *capture basin* correspond to our notions of *sustainable set, sustainable kernel,* and sets of the form $\rightsquigarrow_K A$, but the concepts differ in that we require these sets to be topologically open, to account for possible infinitesimal perturbations. In VP, these and other sets are usually required to be closed instead, and while this has some advantages for proving deep results such as certain convergence properties, it also requires VP to focus on a more restrictive class of systems (differential inclusions and/or Marchaud maps, vector spaces as state spaces) than we do. While our purely topological existence proof only relies on the fact that the sustainable sets form a kernel system, the proof that a viability kernel exists is harder and requires additional smoothness assumptions on the system of possible trajectories.

On the other hand, we have added the distinction between default and alternative trajectories here to be able to talk about the consequences of having to manage a system only temporarily or repeatedly. Consequently, our notion of *shelter* has no counterpart in standard VP, and our notion of *invariance* differs from theirs since it refers to the default dynamics only.

Similarly, our notion of *stable reachability* differs in two important ways from VP's notion of reachability: On the one hand, we require it to be "safe" against infinitesimal perturbations, on the other, we allow a trajectory to need infinite time to reach a target exactly (which does not count as reachable in VP) if it can reach arbitrarily small neighbourhoods of the target in finite time, so that in our theory, asymptotically stable fixed points are reachable via the default dynamics. This difference can easily be seen in a slightly changed version of the main text's Fig. C2 (top-right): Assume $\dot{x} = -r - x^2$ and $\dot{r} \in [-1,0]$, i.e., management can only move to the left. While in our theory, the stable branch is stably reachable from below, it is not so in VP since that takes infinite time.

Despite these differences, algorithms such as the tangent method and the viability kernel algorithm by Frankowska and Quincampoix (1990) are quite helpful in our context, too, and we have the following approximate correspondences: $U \approx$ capture basin of S; $M \approx$ viability kernel of X^+; $U + D \approx$ capture basin of M (this was also called a "resilience basin" in Martin (2004); Rougé et al. (2013)); $E^+ + \Upsilon^+ \approx$ the "shadow" of X^+; and $\Theta \approx$ "invariance kernel" of X^-. In the reachability network of networks, the union of ports and rapids "between" two given ports P, P' (and similarly for harbours and docks) corresponds to what is called a "connection basin" between P and P' in VP.

References

Aubin, J.-P.: Viability theory, Birkhäuser, 2009.

Aubin, J.-P., Bayen, A., and Saint-Pierre, P.: Viability Theory. New Directions, Springer Science & Business Media, 2011.

Bever, J. D.: Soil community feedback and the coexistence of competitors: conceptual frameworks and empirical tests, New Phytologist, 157, 465–473, doi:10.1046/j.1469-8137.2003.00714.x, 2003.

Frankowska, H. and Quincampoix, M.: Viability kernels of differential inclusions with constraints: algorithms and applications, International Institute for Applied Systems Analysis Working Paper, 1990.

Kourtev, P. S., Ehrenfeld, J. G., and Häggblom, M.: Exotic Plant Species Alter the Microbial Community Structure and Function in the Soil, Ecology, 83, 3152–3166, doi:10.1890/0012-9658(2002)083[3152:EPSATM]2.0.CO;2, 2002.

Kulmatiski, A., Heavilin, J., and Beard, K. H.: Testing predictions of a three-species plant-soil feedback model, Journal of Ecology, 99, 542–550, doi:10.1111/j.1365-2745.2010.01784.x, 2011.

Levine, J. M., Pachepsky, E., Kendall, B. E., Yelenik, S. G., and Lambers, J. H. R.: Plant-soil feedbacks and invasive spread., Ecology letters, 9, 1005–14, doi:10.1111/j.1461-0248.2006.00949.x, 2006.

Martin, S.: The cost of restoration as a way of defining resilience: a viability approach applied to a model of lake eutrophication, Ecology And Society, 9, 8, 2004.

Poon, G. T.: The influence of soil feedback and plant traits on competition between an invasive plant and co-occurring native and exotic species, Master's thesis, University of Guelph, 2011.

Read, D. B., Bengough, A. G., Gregory, P. J., Crawford, J. W., Robinson, D., Scrimgeour, C. M., Young, I. M., Zhang, K., and Zhang, X.: Plant Roots Release Phospholipid Surfactants That Modify the Physical and Chemical Properties of Soil, New Phytologist, 157, 315–326, doi:10.1046/j.1469-8137.2003.00665.x, 2003.

Rougé, C., Mathias, J. D., and Deffuant, G.: Extending the viability theory framework of resilience to uncertain dynamics, and application to lake eutrophication, Ecological Indicators, 29, 420–433, doi:10.1016/j.ecolind.2012.12.032, 2013.