Supplement of Earth Syst. Dynam., 13, 1677–1688, 2022
https://doi.org/10.5194/esd-13-1677-2022-supplement


*Supplement of*

# Governing change: a dynamical systems approach to understanding the stability of environmental governance

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# Contents

# Supplementary Methods

## Derivation of Generalized Modeling Scale Parameters

We first define the state variables normalized by their steady state value ($R^*$ is the steady state value for $R$, for example):

$$r := \frac{R}{R^*}, \; x_n := \frac{X_n}{X_n^*}, \; y_m := \frac{Y_m}{Y_m^*}.$$

### Resource Equation

We can then write the normalized functions

$$s(r) := \frac{S(rR^*)}{S^*}$$

and

$$e_n(r, g_{1,n}, \ldots, g_{M,n}) := \frac{E_n(rR^*, G_{1,n}^* g_{1,n}, \ldots, G_{M,n}^* g_{M,n})}{E_n^*},$$

where $g_{m,n}(y_m, F_{1,m,n}x_1, \ldots, F_{K,m,n}x_K) := \dfrac{G_{m,n}(y_m Y_m^*, F_{1,m,n}x_1 X_1^*, \ldots, F_{K,m,n}x_K X_K^*)}{G_{m,n}^*}$.

This allows us to rewrite the equation for $\dot{R}$ in terms of the normalized variables and functions:

$$\dot{r} = \frac{S^*}{R^*}s - \sum_n \frac{E_n^*}{R^*}e_n.$$

We then define the *scale parameters*

$$\phi := \frac{S^*}{R^*} = \sum_n \frac{E_n^*}{R^*}, \; \psi_n := \frac{1}{\phi}\frac{E_n^*}{R^*},$$

and finally rewrite the normalized equation as

$$\dot{r} = \phi\Big(s - \sum_n \psi_n e_n\Big).$$

### Resource User and Non-Resource User Actor Equations

We write the normalized functions

$$b_n(e_n) := \frac{B_n(E_n^* e_n)}{B_n^*}, \; q_n(a_n) := \frac{Q_n(A_n^* a_n)}{Q_n^*},$$
$$c_{k,n}^+(W_{k,n}^+ x_k) := \frac{C_{k,n}(W_{k,n}^+ x_k)}{C_{k,n}^{+*}}, \; c_{k,n}^-(W_{k,n}^- x_k) := \frac{C_{k,n}(W_{k,n}^- x_k)}{C_{k,n}^{-*}},$$
$$u_n(x_n) := \frac{U_n(x_n X_n^*)}{U_n^*}, \; l_n(x_n) := \frac{L_n(x_n X_n^*)}{L_n^*}.$$

where $a_n(r, p_{1,n}, \ldots, p_{M,n})$ and $p_{m,n}(y_m, H_{1,m,n}x_1, \ldots, H_{K,m,n}x_K)$ are defined analogously to $e_n$ and $g_{m,n}$, respectively.

This allows us to rewrite the equation for $\dot{X}_n$ in terms of the normalized variables and functions:

$$\dot{x}_n = \frac{B_n^*}{X_n^*} b_n + \frac{Q_n^*}{X_n^*} q_n + \frac{U_n^*}{X_n^*} u_n + \sum_k \frac{C_{k,n}^{+*}}{X_n^*} c_{k,n}^+ - \sum_k \frac{C_{k,n}^{-*}}{X_n^*} c_{k,n}^- - \frac{L_n^*}{X_n^*} l_n.$$

We then define the *scale parameters*

$$\alpha_n := \frac{B_n^*}{X_n^*} + \frac{Q_n^*}{X_n^*} + \frac{U_n^*}{X_n^*} + \sum_k \frac{C_{k,n}^{+*}}{X_n^*} = \sum_k \frac{C_{k,n}^{-*}}{X_n^*} + \frac{L_n^*}{X_n^*},$$

$$\beta_n := \frac{1}{\alpha_n} \frac{B_n^*}{X_n^*},\ \ \widehat{\beta}_n := \frac{1}{\alpha_n} \frac{Q_n^*}{X_n^*},\ \ \overline{\beta}_n := \frac{1}{\alpha_n} \frac{U_n^*}{X_n^*},\ \ \widetilde{\beta}_n := \frac{1}{\alpha_n} \sum_k \frac{C_{k,n}^{+*}}{X_n^*},\ \ \sigma_{k,n} := \frac{1}{\alpha_n \widetilde{\beta}_n} \frac{C_{k,n}^{+*}}{X_n^*},$$

$$\overline{\eta}_n := \frac{1}{\alpha_n} \frac{L_n^*}{X_n^*},\ \ \eta_n := \frac{1}{\alpha_n} \sum_k \frac{C_{k,n}^{-*}}{X_n^*},\ \ \lambda_{k,n} = \frac{1}{\alpha_n \eta_n} \frac{C_{k,n}^{-*}}{X_n^*}.$$

Finally, we rewrite the normalized equation as

$$\dot{x}_n = \alpha_n \Big( \beta_n b_n + \widehat{\beta}_n q_n + \overline{\beta}_n u_n + \widetilde{\beta}_n \sum_k \sigma_{k,n} c_{k,n}^+ - \eta_n \sum_k \lambda_{k,n} c_{k,n}^- - \overline{\eta}_n l_n \Big).$$

### Governance Institution Equations

We write the normalized function

$$i_m^+(y_m, K_{1,m}^+ x_1, \ldots, K_{N,m}^+ x_N) := \frac{I_m^+(y_m Y^*, K_{1,m}^+ x_1 X_1^*, \ldots, K_{N,m}^+ x_N X_N^*)}{I_m^{+*}},$$

and likewise for $i_m^-$.

This allows us to rewrite the equation for $\dot{Y}_m$ in terms of the normalized variables and functions as

$$\dot{y}_m = \frac{I_m^{+*}}{Y^*} i_m^+ - \frac{I_m^{-*}}{Y^*},$$

which leads us to define the scale parameter

$$\mu_m := \frac{I_m^{+*}}{Y^*} = \frac{I_m^{-*}}{Y^*}.$$

Finally, we rewrite the normalized equation as

$$\dot{y}_m = \mu_m \left( i_m^+ - i_m^- \right).$$

## Jacobian and Exponent Parameters

We find the relevant *exponent parameters* by looking at the corresponding entries of the Jacobian.

### From the Resource Equation

$$\frac{\partial \dot{r}}{\partial r} = \phi \left( \frac{\partial s}{\partial r} - \sum_n \psi_n \frac{\partial e_n}{\partial r} \right)$$

$$\frac{\partial \dot{r}}{\partial x_i} = -\phi \sum_n \psi_n \sum_m \frac{\partial e_n}{\partial g_{m,n}} \cdot \frac{\partial g_{m,n}}{\partial (F_{i,m,n} x_i)} \cdot F_{i,m,n}$$

$$\frac{\partial \dot{r}}{\partial y_m} = -\phi \sum_n \psi_n \frac{\partial e_n}{\partial g_{m,n}} \cdot \frac{\partial g_{m,n}}{\partial y_m}$$

### From the Resource User and Non-Resource User Actor Equations

$$\frac{\partial \dot{x}_n}{\partial r} = \alpha_n \left( \beta_n \frac{\partial b_n}{\partial e_n} \cdot \frac{\partial e_n}{\partial r} + \widehat{\beta}_n \frac{\partial q_n}{\partial a_n} \cdot \frac{\partial a_n}{\partial r} \right)$$

For $i \neq n$:

$$\begin{aligned}\frac{\partial \dot{x}_n}{\partial x_i} = \alpha_n \Bigg( & \beta_n \frac{\partial b_n}{\partial e_n} \cdot \sum_m \frac{\partial e_n}{\partial g_{m,n}} \cdot \frac{\partial g_{m,n}}{\partial (F_{i,m,n} x_i)} \cdot F_{i,m,n} \\ & + \widehat{\beta}_n \frac{\partial q_n}{\partial a_n} \cdot \sum_m \frac{\partial a_n}{\partial p_{m,n}} \cdot \frac{\partial p_{m,n}}{\partial (H_{i,m,n} x_i)} \cdot H_{i,m,n} \\ & + \widetilde{\beta}_n \sigma_{i,n} \frac{\partial c_{i,n}^+}{\partial (W_{i,n}^+ x_i)} W_{i,n}^+ - \eta_n \lambda_{i,n} \frac{\partial c_{i,n}^-}{\partial (W_{i,n}^- x_i)} W_{i,n}^- \Bigg)\end{aligned}$$

For $i = n$:

$$\begin{aligned}\frac{\partial \dot{x}_n}{\partial x_n} = \alpha_n \Bigg( & \beta_n \frac{\partial b_n}{\partial e_n} \cdot \sum_m \frac{\partial e_n}{\partial g_{m,n}} \cdot \frac{\partial g_{m,n}}{\partial (F_{n,m,n} x_n)} \cdot F_{n,m,n} \\ & + \widehat{\beta}_n \frac{\partial q_n}{\partial a_n} \cdot \sum_m \frac{\partial a_n}{\partial p_{m,n}} \cdot \frac{\partial p_{m,n}}{\partial (H_{n,m,n} x_n)} \cdot H_{n,m,n} \\ & + \overline{\beta}_n \frac{\partial u_n}{\partial x_n} - \overline{\eta}_n \frac{\partial l_n}{\partial x_n} \Bigg)\end{aligned}$$

$$\frac{\partial \dot{x}_n}{\partial y_m} = \alpha_n \left( \beta_n \frac{\partial b_n}{\partial e_n} \cdot \frac{\partial e_n}{\partial g_{m,n}} \cdot \frac{\partial g_{m,n}}{\partial y_m} + \widehat{\beta}_n \frac{\partial q_n}{\partial a_n} \cdot \frac{\partial a_n}{\partial p_{m,n}} \cdot \frac{\partial p_{m,n}}{\partial y_m} \right)$$

### From the Governance Institution Equations

$$\frac{\partial \dot{y}_m}{\partial r} = 0$$

$$\frac{\partial \dot{y}_m}{\partial x_i} = \mu_m \left[ \frac{\partial i_m^+}{\partial (K_{i,m}^+ x_i)} K_{i,m}^+ - \frac{\partial i_m^-}{\partial (K_{i,m}^- x_i)} K_{i,m}^- \right]$$

$$\frac{\partial \dot{y}_m}{\partial y_m} = \mu_m \left[ \frac{\partial i_m^+}{\partial y_m} - \frac{\partial i_m^-}{\partial y_m} \right]$$

For $j' \neq m$:

$$\frac{\partial \dot{y}_m}{\partial y_{j'}} = 0$$

## Derivation of Objective Function Gradient

At equilibrium, the equation

$$\frac{d}{d\mathfrak{p}} \begin{pmatrix} r^* \\ x_n^* \\ y_m^* \end{pmatrix} = -J^{-1} \begin{pmatrix} \frac{\partial \dot{r}}{\partial \mathfrak{p}} \\ \frac{\partial \dot{x}_n}{\partial \mathfrak{p}} \\ \frac{\partial \dot{y}_m}{\partial \mathfrak{p}} \end{pmatrix}$$

describes how the steady state changes with respect to a strategy parameter $\mathfrak{p}$. The following sections show the calculation of the right-hand side of this equation for each of the strategy parameters.

## Calculation of Right-Hand Side

### Calculations for $F_{k,m,n}$

$$\frac{\partial \dot{r}}{\partial F_{k,m,n}} = -\phi\psi_n \frac{\partial e_n}{\partial g_{m,n}} \cdot \frac{\partial g_{m,n}}{\partial (F_{k,m,n} x_k)}$$

$$\frac{\partial \dot{x}_i}{\partial F_{k,m,n}} = \begin{cases} \alpha_n \beta_n \frac{\partial b_n}{\partial e_n} \cdot \frac{\partial e_n}{\partial g_{m,n}} \cdot \frac{\partial g_{m,n}}{\partial (F_{k,m,n} x_k)} & \text{if } i = n \\ 0 & \text{if } i \neq n \end{cases}$$

$$\frac{\partial \dot{y}_j}{\partial F_{k,m,n}} = 0$$

### Calculations for $H_{k,m,n}$

$$\frac{\partial \dot{r}}{\partial H_{k,m,n}} = 0$$

$$\frac{\partial \dot{x}_i}{\partial H_{k,m,n}} = \begin{cases} \alpha_n \widehat{\beta}_n \frac{\partial q_n}{\partial a_n} \cdot \frac{\partial a_n}{\partial p_{m,n}} \cdot \frac{\partial p_{m,n}}{\partial (H_{k,m,n} x_k)} & \textbf{if } i = n \\ 0 & \textbf{if } i \neq n \end{cases}$$

$$\frac{\partial \dot{y}_j}{\partial H_{k,m,n}} = 0$$

### Calculations for $W^+_{k,n}$ and $W^-_{k,n}$

$$\frac{\partial \dot{r}}{\partial W^+_{k,n}} = 0$$

$$\frac{\partial \dot{x}_i}{\partial W^+_{k,n}} = \begin{cases} \alpha_n \widetilde{\beta}_n \sigma_{k,n} \frac{\partial c^+_{k,n}}{\partial (W^+_{k,n} x_k)} & \text{if } i = n \\ 0 & \text{if } i \neq n \end{cases}$$

$$\frac{\partial \dot{y}_j}{\partial W^+_{k,n}} = 0$$

$$\frac{\partial \dot{r}}{\partial W^-_{k,n}} = 0$$

$$\frac{\partial \dot{x}_i}{\partial W^-_{k,n}} = \begin{cases} -\alpha_n \eta_n \lambda_{k,n} \frac{\partial c^-_{k,n}}{\partial (W^-_{k,n} x_k)} & \text{if } i = n \\ 0 & \text{if } i \neq n \end{cases}$$

$$\frac{\partial \dot{y}_j}{\partial W^-_{k,n}} = 0$$

**Calculations for $K^+_{k,m}$ and $K^-_{k,m}$**

$$\frac{\partial \dot{r}}{\partial K^+_{k,m}} = 0$$
$$\frac{\partial \dot{x}_i}{\partial K^+_{k,m}} = 0$$
$$\frac{\partial \dot{y}_j}{\partial K^+_{k,m}} = \begin{cases} \mu_m \dfrac{\partial i^+_m}{\partial (K^+_{k,m} x_k)} & \text{if } j = m \\ 0 & \text{if } j \neq m \end{cases}$$

$$\frac{\partial \dot{r}}{\partial K^-_{k,m}} = 0$$
$$\frac{\partial \dot{x}_i}{\partial K^-_{k,m}} = 0$$
$$\frac{\partial \dot{y}_j}{\partial K^-_{k,m}} = \begin{cases} -\mu_m \dfrac{\partial i^-_m}{\partial (K^-_{k,m} x_k)} & \text{if } j = m \\ 0 & \text{if } j \neq m \end{cases}$$

## Calculating how objective functions change with each parameter

### Extraction

We have

$$\frac{de_n}{dF_{l,j,n}} = \frac{\partial e_n}{\partial r}\frac{\partial r^*}{\partial F_{l,j,n}} + \sum_m \left( \frac{\partial e_n}{\partial g_{m,n}}\frac{\partial g_{m,n}}{\partial y_m}\frac{\partial y^*_m}{\partial F_{l,j,n}} + \sum_k \frac{\partial e_n}{\partial g_{m,n}}\frac{\partial g_{m,n}}{\partial (F_{k,m,n} x_k)}\frac{\partial x^*_k}{\partial F_{l,j,n}} \cdot F_{k,m,n} \right) + \frac{\partial e_n}{\partial g_{j,n}}\frac{\partial g_{j,n}}{\partial (F_{l,j,n} x_l)}.$$

For any other effort allocation parameter $\mathfrak{p}$, including $\mathfrak{p} = F_{l,j,i}$ when $i \neq n$, we can use the general formula

$$\frac{de_n}{d\mathfrak{p}} = \frac{\partial e_n}{\partial r}\frac{\partial r^*}{\partial \mathfrak{p}} + \sum_m \left( \frac{\partial e_n}{\partial g_{m,n}}\frac{\partial g_{m,n}}{\partial y_m}\frac{\partial y^*_m}{\partial \mathfrak{p}} + \sum_k \frac{\partial e_n}{\partial g_{m,n}}\frac{\partial g_{m,n}}{\partial (F_{k,m,n} x_k)}\frac{\partial x^*_k}{\partial \mathfrak{p}} \cdot F_{k,m,n} \right).$$

### Access

We have

$$\frac{da_n}{dH_{l,j,n}} = \frac{\partial a_n}{\partial r}\frac{\partial r^*}{\partial H_{l,j,n}} + \sum_m \left( \frac{\partial a_n}{\partial p_{m,n}}\frac{\partial p_{m,n}}{\partial y_m}\frac{\partial y^*_m}{\partial H_{l,j,n}} + \sum_k \frac{\partial a_n}{\partial p_{m,n}}\frac{\partial p_{m,n}}{\partial (H_{k,m,n} x_k)}\frac{\partial x^*_k}{\partial H_{l,j,n}} \cdot H_{k,m,n} \right) + \frac{\partial a_n}{\partial p_{j,n}}\frac{\partial p_{j,n}}{\partial (H_{l,j,n} x_l)}.$$

For any other effort allocation parameter $\mathfrak{p}$, we can use the general formula

$$\frac{da_n}{d\mathfrak{p}} = \frac{\partial a_n}{\partial r}\frac{\partial r^*}{\partial \mathfrak{p}} + \sum_m \left( \frac{\partial a_n}{\partial p_{m,n}}\frac{\partial p_{m,n}}{\partial y_m}\frac{\partial y^*_m}{\partial \mathfrak{p}} + \sum_k \frac{\partial a_n}{\partial p_{m,n}}\frac{\partial p_{m,n}}{\partial (H_{k,m,n} x_k)}\frac{\partial x^*_k}{\partial \mathfrak{p}} \cdot H_{k,m,n} \right).$$

## Parameter Values and Ranges

Parameters are derived from the Generalized Modeling approach described above.

| Parameter | Interpretation | Range | Value |
|---|---|---|---|
| **Scale Parameters** | | | |
| $\phi$ | Rate of turnover in the resource, or inverse of characteristic time scale of resource | 0 to 1 | |
| $\psi_n$ | Share of extraction of resource by user $n$ | 0 to 1, $\sum_n \psi_n = 1$ | |
| $\alpha_n$ | Rate of turnover in the capacity of user $n$ | 0 to 1 | |
| $\beta_n$ | Share of actor $n$ capacity gain in response to resource extraction | $\beta_n + \widehat{\beta}_n + \widetilde{\beta}_n + \overline{\beta}_n = 1$ | |
| $\widehat{\beta}_n$ | Share of actor $n$ capacity gain in response to resource access conditions | $\beta_n + \widehat{\beta}_n + \widetilde{\beta}_n + \overline{\beta}_n = 1$ | |
| $\widetilde{\beta}_n$ | Share of actor $n$ capacity gain from collaborations | $\beta_n + \widehat{\beta}_n + \widetilde{\beta}_n + \overline{\beta}_n = 1$ | |
| $\overline{\beta}_n$ | Share of actor $n$'s capacity gain from "natural" gain (non-resource users only) | $\beta_n + \widehat{\beta}_n + \widetilde{\beta}_n + \overline{\beta}_n = 1$ | |
| $\sigma_{k,n}$ | Share of actor $n$'s collaboration gain from collaborating with actor $k$ | 0 to 1 | |
| $\eta_n$ | Share of actor $n$'s loss in capacity due to direct undermining by other actors | $1 - \overline{\eta}_n$ | |
| $\lambda_{k,n}$ | Share of actor $n$'s loss from being undermined by other actors attributed to actor $k$ | 0 to 1 | |
| $\overline{\eta}_n$ | Share of actor $n$'s loss in capacity due to "natural" decay | $1 - \eta_n$ | |
| $\mu_m$ | Rate of turnover in decision center $m$'s capacity | 0 to 1 | |
| **Exponent Parameters** | | | |
| $\frac{\partial s}{\partial r}$ | Sensitivity of resource regeneration to resource state | $-1$ to 1 | $-0.5$ |
| $\frac{\partial e_n}{\partial r}$ | Sensitivity of extraction by user $n$ to resource state | 1 to 2 | 1.5 |
| $\frac{\partial e_n}{\partial g_{m,n}}$ | Sensitivity of extraction by user $n$ to intervention by decision center $m$ (effectiveness of intervention) | $-1$ to 1 | - |
| $\frac{\partial g_{m,n}}{\partial (F_{i,m,n} x_i)}$ | Sensitivity of intervention in user $n$'s extraction by decision center $m$ to actions by actor $i$ (effectiveness of actors' support/resistance) | 0 to 2 | 1 |
| $\frac{\partial g_{m,n}}{\partial y_m}$ | Sensitivity of extraction intervention by decision center $m$ to their own capacity | 0 to 2 | 1 |
| $\frac{\partial p_{m,n}}{\partial y_m}$ | Sensitivity of resource access intervention by decision center $m$ to their own capacity | 0 to 2 | 1 |
| $\frac{\partial b_n}{\partial e_n}$ | Sensitivity of user $n$'s gain in capacity based on extraction to the amount of extraction | $-1$ to 1 | 0.5 |
| $\frac{\partial a_n}{\partial r}$ | Sensitivity of access by user $n$ to resource state | 0 to 2 | 1 |
| $\frac{\partial q_n}{\partial a_n}$ | Sensitivity of user $n$'s gain in capacity based on resource access to the level of resource access | $-1$ to 1 | 0.5 |
| $\frac{\partial a_n}{\partial p_{m,n}}$ | Effectiveness of intervention $p$ by decision center $m$ in changing access for resource user $n$ | $-1$ to 1 | - |

| | | | |
|---|---|---|---|
| $\frac{\partial p_{m,n}}{\partial(H_{i,m,n}x_i)}$ | Sensitivity of intervention by decision center $m$ to actions by actor $i$ (effectiveness of actors' support/resistance) | 0 to 2 | 1 |
| $\frac{\partial c^+_{i,n}}{\partial(W^+_{i,n}x_i)}$ | Sensitivity of actor $n$'s gain from collaboration to actor $i$'s collaboration efforts | 0 to 2 | 1 |
| $\frac{\partial c^-_{i,n}}{\partial(W^-_{i,n}x_i)}$ | Sensitivity of actor $n$'s loss in capacity to other actor $i$'s efforts to undermine them | 0 to 2 | 1 |
| $\frac{\partial l_n}{\partial x_n}$ | Sensitivity of actor $n$'s "natural" decay in capacity $l$ to their own capacity | 0.5 to 1 | 1 |
| $\frac{\partial u_n}{\partial x_n}$ | Sensitivity of non-resource user actor $n$'s self-growth in capacity to their own capacity | 0 to 1 | 0.5 |
| $\frac{\partial i^+_m}{\partial(K^+_{i,m}x_i)}$ | Sensitivity of decision center $m$'s gain in capacity to actor $i$'s actions; likewise for $\frac{\partial i^-_m}{\partial(K^-_{i,m}x_i)}$ | 0 to 2 | 1 |
| $\frac{\partial i^+_m}{\partial y_m}$ | Sensitivity of decision center $m$'s gain in capacity to their own capacity | 0 to 1 | 0.5 |
| $\frac{\partial i^-_m}{\partial y_m}$ | Sensitivity of decision center $m$'s loss in capacity to their own capacity | 0 to 1 | 1 |

# Supplementary Figures

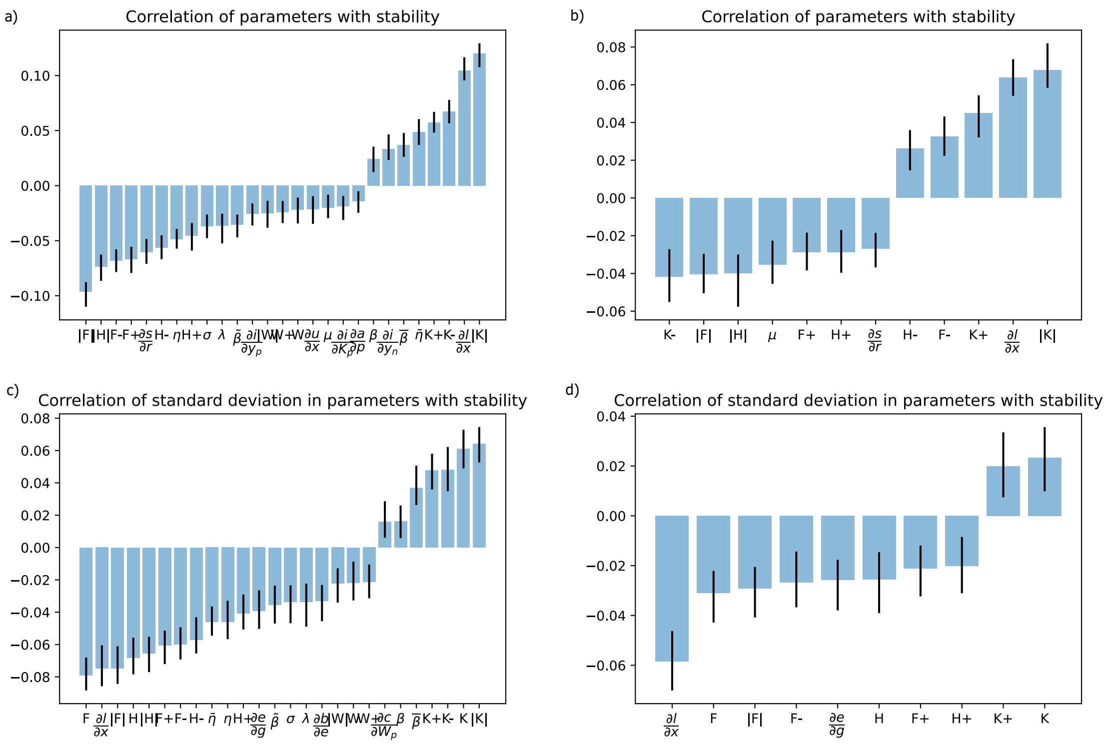


Figure S1: Correlation results including all forms of strategy parameters and all significant parameters. The inclusion of the different forms of strategy parameters allows for concluding that stability depends on the magnitude of effort allocated to the strategies rather than the sign or direction of the effort.

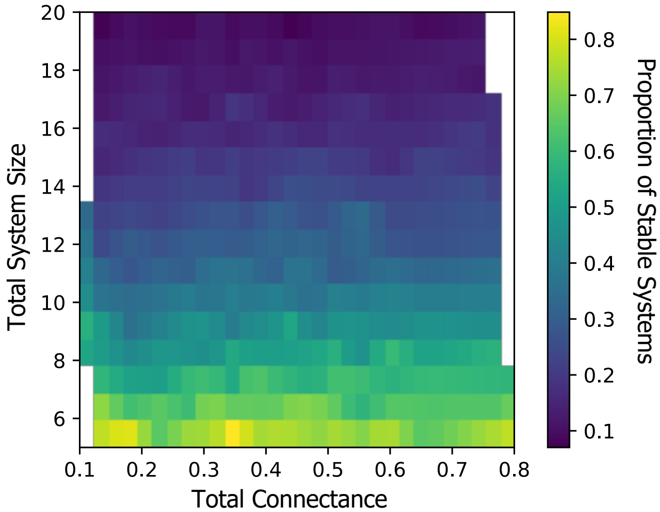


Figure S2: Effect of system size (number of actors and decision centers) and connectance on stability. The connectance shown is the total connectance, which is computed after the experiment rather than set beforehand due to the dependence of the connectance on actors' computed strategies. As a result, there is no data for some combinations of connectance and size.

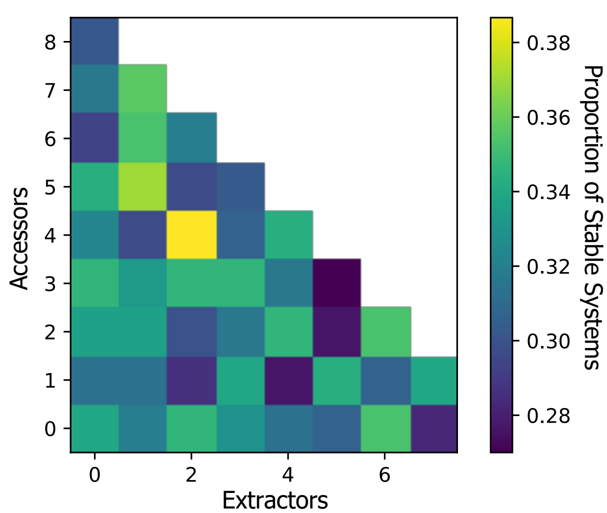


Figure S3: Effect of different types of resource users (extractors, accessors, and combined extractors and accessors) on stability. The color represents the proportion of stable systems for a given system composition. The total system size is 10, with 8 resource users and 2 decision centers. The proportion of extractors as compared to accessors or combined extractors and accessors has no effect on stability.